Problems related to

"Projective Planes and Beyond"
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Lecture 1. Projective planes

In today's lecture we gave the axiomatic approach for a projective plane $\mathbb{P}^2$, and we now recall the axioms we used. Remember that $\mathbb{P}^2$ consists of a set $\mathfrak{P}$ of points and a collection $\mathfrak{L}$ of subsets called *lines*, satisfying these axioms (from Moorhouse):

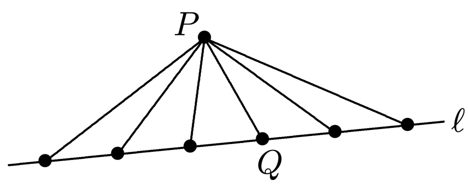
(P1) Given any two points P, Q there is a unique line $\overline{PQ}$ containing both P and Q.
(P2) Given any two lines ℓ, m there is a unique point P lying both on ℓ and on m (i.e. any two lines meet in a point).
(P3) There exist four points such that no three are collinear.

1. In the lecture we mentioned an exercise. Let ℓ, m be distinct lines in $\mathbb{P}^2$. Using the axioms (P1), (P2) and (P3), show that there exists a point $P \notin [\ell] \cup [m]$, where $[\ell]$ (resp. $[m]$) denotes the set of points lying on ℓ (resp. m).

2. We defined $\mathbb{P}^2_{\mathbb{R}}$ by starting with the Euclidean plane and adding a collection of points at infinity, one for each family of parallel lines. We called the set of new points at infinity *the line at infinity* $\mathbb{P}^1_{\mathbb{R}}$ (so in particular, we view this set as a line). In this way, it's obvious that
$$\mathbb{P}^2_{\mathbb{R}} = \mathbb{R}^2 \cup \mathbb{P}^1_{\mathbb{R}}.$$
Verify that the resulting $\mathfrak{P}$ and $\mathfrak{L}$ satisfy the axioms for a $\mathbb{P}^2$.

3. In the lecture we claimed:

   If $P \notin \ell$ then an obvious bijection $[\ell] \to [P]$ consists of mapping each point $Q \in \ell$ to the line PQ.

   Verify that this gives a bijection, using the axioms.

4. From the lecture:

   Let ℓ and m be distinct lines. Using the axioms, check that it follows that there exists a point $P \notin [\ell] \cup [m]$.

5. We saw that $\mathbb{P}^2_{\mathbb{R}}$ can also be interpreted as the set of lines through the origin in $\mathbb{R}^3$. Replacing $\mathbb{R}$ by any field k, show that $\mathbb{P}^2_k = k^2 \cup \mathbb{P}^1_k$ where the latter is defined as the set of lines through the origin in k^2.

6. In the lecture we saw a picture of the Fano plane, a finite projective plane with 7 elements constructed as the classical projective plane over $k = \mathbb{Z}_2$. Draw an analogous picture of the finite plane over $k = \mathbb{Z}_3$. [If you give up, see Moorhouse's book, page 36.]

7. There are also higher dimensional projective spaces. Here is one way we can make axioms for $\mathbb{P}^3$. We will say that:

$\mathbb{P}^3$ consists of a set $\mathfrak{P}$ of *points*, a collection $\mathfrak{L}$ of *lines* and a collection $\mathfrak{H}$ of *planes*,

satisfying the following axioms.

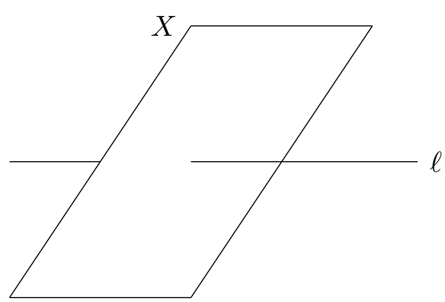
(S1) Any two distinct points lie on exactly one line.
(S2) Any two distinct planes meet in exactly one line.
(S3) If a plane contains a line, it contains all the elements of that line.
(S4) Two distinct lines meet in a point if and only if they lie in a common plane.
(S5) There exists a set of five points, of which no four lie in a common plane.
(S6) Every line contains at least three points.
(S7) if X is a plane and P_1, P_2 are points of X then X contains the entire line spanned by P_1 and P_2 (whose existence is guaranteed by (S1)).

For each of the following problems you can refer to any earlier problem in addition to using the axioms. You can do this regardless of whether you were able to prove the earlier problem or not.

(a) Prove that three noncollinear points lie on a unique plane. (Be sure to prove uniqueness as part of your answer.)

(b) Given any line ℓ and any point P not on ℓ, prove that there exists a unique plane containing both P and ℓ.

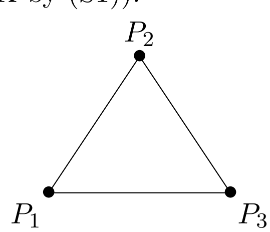
(c) Let X be any plane and let ℓ be any line not contained in X. Prove that X must meet ℓ in exactly one point.



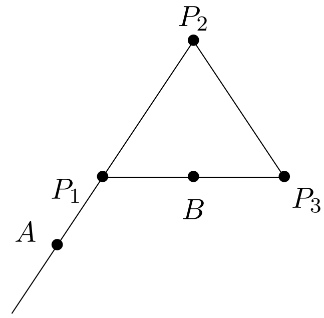
(d) If X is a plane, show that it is a $\mathbb{P}^2$; i.e. show that axioms (P1), (P2), (P3) hold.

(e) Prove that every line meeting two sides of a triangle, but none of its vertices, must also meet the third side. More precisely:

Consider the triangle P_1, P_2, P_3. We'll interpret this as the data consisting of the three points P_1, P_2, P_3 together with the corresponding three lines that they span pairwise, which we'll denote $\overline{P_1P_2}$, $\overline{P_1P_3}$, $\overline{P_2P_3}$ (this is OK by (S1)).



Choose a third point, A, on $\overline{P_1P_2}$ and a third point, B, on $\overline{P_1P_3}$ (this is ok by (S6)):



Then prove that the line $\overline{AB}$ has to meet the line $\overline{P_2P_3}$ in a point.

(f) If the $\mathbb{P}^3$ is finite, show that any two lines of $\mathbb{P}^3$ have the same number of elements, which we'll still call $d+1$. (Note that the two lines don't necessarily meet in a point, so they're not necessarily in the same plane. Note also that we have not assumed that $\mathbb{P}^3$ is built up from any field.)

(g) Show that any two planes contain the same number of elements. What is that number (in terms of the integer d in Problem 7f)? Explain your answer. Again, feel free to use earlier problems.

(h) In terms of d (as in Problem 7f), how many points are in $\mathbb{P}^3$? Explain your answer using the axioms and previous problems.

Lecture 2. A first look at Hilbert functions

1. Recall the construction, given in the lecture, that produced a pure O-sequence from a given projective plane $\mathbb{P}^2$. First of all, each line contains $d+1$ points, where d is the *order* of the given $\mathbb{P}^2$. (Different projective planes have different orders in general. Here we assume we have picked a specific $\mathbb{P}^2$.) Furthermore, $\mathbb{P}^2$ contains
$$q = d^2 + d + 1$$
points (this is the definition of q).

   To each point of $\mathbb{P}^2$ we associate a unique variable, x_i, with $1 \le i \le q$. We let $\mathcal{M}_{d+1}$ be the squarefree monomials of degree $d+1$ corresponding to points on the same line. We let $\mathcal{M}_d$ be the (squarefree) monomials of degree d that divide at least one monomial in $\mathcal{M}_{d+1}$, and we continue to define $\mathcal{M}_{d-1}, \mathcal{M}_{d-2}, \ldots, \mathcal{M}_1, \mathcal{M}_0$ in the same way.

   We define $h_j = |\mathcal{M}_j|$ for $0 \le j \le d+1$.

   Consider the sequence
$$\left(1, q, q\binom{d+1}{2}, q\binom{d+1}{3}, \ldots, q\binom{d+1}{d}, q\binom{d+1}{d+1}\right). \tag{1}$$

   (a) Prove that the sequence (1) is exactly the sequence $(1, h_1, h_2, \ldots, h_{d+1})$, using the axioms of a projective plane.

   (b) Conversely, assume we are given the sequence (1), viewed just as a sequence of integers. Prove that if it is a pure O-sequence (i.e. if there exists a set of squarefree monomials $\mathcal{M}_{d+1}$ which generates the other $\mathcal{M}_j$ as above), then there exists a projective plane of order d.

2. For the projective plane $\mathbb{P}^2_{\mathbb{Z}_3}$, you can use the previous exercise to write the pure O-sequence, but this problem asks you to write out the sets $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_4$. Recall that $q\binom{d+1}{2} = \binom{q}{2}$.

3. Prove that if $R = k[x_0, x_1, \dots, x_n]$ then

$$\dim_k R_t = \binom{n+t}{n}$$

with basis given by the monomials of degree t.

4. Prove that for an ideal $I \subset k[x_0, \dots, x_n]$ the following are equivalent:

   (a) If $f \in I$ then the homogeneous components of f are also in I.

   (b) The ideal I has a generating set consisting of homogeneous polynomials.

   The ideal I is said to be *homogeneous* if either of the (equivalent) conditions (a) or (b) hold.

5. Let k be a field. If I is a homogeneous ideal in $R = k[x_0, \dots, x_n]$ and t is a positive integer, let I_t be the set of homogeneous polynomials in I of degree t. Prove that I_t is a vector space over the field k. (Note that we fix k, but that it is an arbitrary field.)

6. Explain why the properties of an ideal are what you need for coset multiplication to be well-defined:

$$(f + I)(g + I) = fg + I.$$

7. Recall that for a homogeneous ideal $I \subset R = k[x_0, \dots, x_n]$, the Hilbert function of $A = R/I$ is

$$h_A(t) = \dim A_t = \dim R_t - \dim I_t = \binom{n+t}{n} - \dim I_t.$$

   Find the Hilbert function of R/I where $R = k[x, y, z]$ and $I = \langle x^3, y^3, z^3, xyz \rangle$. (We will revisit this example later.)

8. Let $\underline{h} = (1, 4, 5, 6, 7, \dots, t + 3, \dots)$ (where the $t + 3$ occurs in degree t and the function continues to grow linearly after that as well). Find an example of a variety in $\mathbb{P}^3$ with this Hilbert function. Do this as follows.

   (a) Start with the Hilbert function $(1, 2, 3, 4, 5, \dots, t+1, \dots)$. Show that this is the Hilbert function of a line in $\mathbb{P}^3$.

   (b) Now focus on what to do about that 4 in degree 1 of $\underline{h}$, i.e. how do you go from the line, with Hilbert function $(1, 2, 3, 4, \dots)$, to the desired Hilbert function.

Lecture 3. Who or what lives in projective space?

1. Prove that if k is an infinite field and $f \in k[x_1, \dots, x_n]$ then the following are equivalent:

   (a) f is the zero polynomial.

   (b) The evaluation function $f : k^n \to k$, defined by $f(P) = f(b_1, \dots, b_n)$ for $P = (b_1, \dots, b_n) \in k^n$, is the zero function. (I.e. f, evaluated at any point of k^n, vanishes.)

2. Now instead consider a finite field. Let p be a prime and consider the field $\mathbb{Z}_p$. Give an example of a polynomial $f \in \mathbb{Z}_p[x, y]$ for which $f : \mathbb{Z}_p^2 \to \mathbb{Z}_p$ vanishes at **all but one** point of $\mathbb{Z}_p^2$. (Specifically, it has to fail to vanish at one and only one point of $\mathbb{Z}_p^2$.) Be sure to prove why your example works – it's not enough to just give the polynomial. [Hint: Fermat's Little Theorem.]

3. If k is an infinite field, prove that the phenomenon in Problem 2 can't happen. That is, prove that if $f \in k[x,y]$ and $f(x,y) = 0$ when evaluated at every point of k^2 except one specific point (a,b) then we must also have $f(a,b) = 0$. (Make sure to indicate the relevance of the assumption that k is infinite.)

4. Let $R = k[x_1, \dots, x_n]$ and let I and J be ideals. Define

$$I : J = \{f \in R \mid fg \in I \text{ for all } g \in J\}.$$

   (a) Prove that if I is radical then so is $I : J$.

   (b) Give an example to show that if I is not radical then $I : J$ need not be radical. (So you have to find a suitable I and a suitable J.)

5. Prove that if I is a homogeneous ideal then so is $\sqrt{I}$.

6. If I is a prime ideal, show that $\sqrt{I} = I$.

7. We mentioned that finite unions and arbitrary intersections of projective varieties are again projective varieties. Give an example to show that arbitrary unions of projective varieties are not necessarily again projective varieties.

8. Along the same lines, find the smallest closed set in $\mathbb{P}^2_{\mathbb{R}}$ containing the set

$$T = \{[1, p, q] \in \mathbb{P}^2_{\mathbb{R}} \mid p \text{ and } q \text{ are prime numbers}\}.$$

9. Let $Z \subset \mathbb{P}^n_k$ be a finite set of r points. Let $I = \mathbb{I}(Z)$ and let h_Z be the Hilbert function of R/I. For this problem it will be helpful to remember that R/I is a graded ring, so for any integer t and any linear form L there's a multiplication map

$$\times L : [R/I]_t \to [R/I]_{t+1}.$$

   (a) Prove that for $t \gg 0$, $h_Z(t) = r$. [Hint: let L be general and remember that

$$\mathbb{I}(Z) = \{f \in R \mid f(P) = 0 \text{ for all } P \in Z\}.$$

   Think about injectivity.]

   (b) Prove that h_Z is strictly increasing until it takes the value r, and then it has the value r forever afterwards.

10. Let $Z \subset \mathbb{P}^2_{\mathbb{R}}$ be a set of six points with Hilbert function $\underline{h} = (1, h_1, h_2, h_3, \dots) = (1, 4, 5 \dots)$.

   (a) Prove that h_3 must be equal to 6. [Hint: see Exercise #9b in this section.]

   (b) If I were not necessarily equal to the ideal of a finite set of points but instead were any homogeneous ideal, what are the possible values for h_4? Note that the given information only goes up to h_2 – ignore your answer to #10a.

11. (a) Prove that in the polynomial ring $R = \mathbb{C}[x, y, z, w]$ (or indeed, in any polynomial ring), the homogeneous polynomials of fixed degree d form a vector space over $\mathbb{C}$. (Here the specific field is irrelevant.)

   (b) Prove that in $R = \mathbb{C}[x, y, z, w]$ there is a 10-dimensional vector space of homogeneous polynomials of degree 2. (There is a 1-line answer using an earlier problem.)

(c) Let λ_1 be the line defined by the ideal $\langle x, y\rangle$. Let λ_2 be the line defined by the ideal $\langle z, w\rangle$. Let λ_3 be the line defined by the ideal $\langle x-w, y-z\rangle$. Find a homogeneous polynomial of degree 2 vanishing on these three lines. Equivalently, find a quadric surface containing them.

(d) Let $\lambda_1, \lambda_2, \lambda_3$ be disjoint lines in $\mathbb{P}^3_{\mathbb{C}}$ and let $C = \lambda_1 \cup \lambda_2 \cup \lambda_3$. Prove that there is a unique quadric surface containing C. Equivalently, there is a unique homogeneous polynomial, up to scalar multiplication, vanishing on the three lines. One way to do this is via the following steps:

(i) Find (with proof) 9 points on C with the property that a homogeneous polynomial of degree 2 vanishes on all of C if and only if it vanishes on the 9 points.

(ii) Prove that if you remove any one of the 9 points then there is a homogeneous polynomial of degree 2 vanishing on the remaining 8 points but *not* on all of C.

(iii) Conclude from #11(d)i and #11(d)ii that C imposes exactly 9 independent conditions on the 10-dimensional vector space mentioned in #11b, meaning that the linear subspace of homogeneous polynomials of degree 2 vanishing on C is $10-9=1$ dimensional.

(iv) Why does the conclusion in #11(d)iii mean that there is a unique quadric surface containing C?

Lecture 4. Lefschetz properties

1. Show that the characteristic of a field has to be 0 or a prime number.
2. In any characteristic, let R/I be a **monomial** algebra (i.e. a quotient R/I, where I is generated by monomials).

   (a) Prove that the homomorphism
   $$\times L : [R/I]_t \to [R/I]_{t+1}$$
   has maximal rank for **general** $L \in R_1$ if and only if $\times(x_1 + \cdots + x_n)$ has maximal rank. [Hint: think change of variables.]

   (b) Give an example to show this is not necessarily true without the assumption that the algebra be monomial.
3. Let $R = \mathbb{C}[x, y]$. Let $I = \langle x^a, y^b\rangle$. Prove that R/I **has** the WLP. Then prove that it **has** the SLP. Keep in mind #2a.
4. Let $R = \mathbb{R}[x, y, z]$ and let $I = \langle x^3, y^3, z^3, xyz\rangle$. Prove that R/I **does not** have the WLP. Keep in mind #2a.
5. Let X be a subvariety of $\mathbb{P}^n_{\mathbb{C}}$. Let L be a general linear form. Show that $R/\mathbb{I}(X)$ has the WLP. Indeed, show that it has the SLP. Keep in mind #2b.
6. (Non-Lefschetz locus)

   (a) Let $I = \langle x^3, y^3\rangle \subset \mathbb{C}[x, y] = R$. Let $A = R/I$.

   (i) Find the Hilbert function of A. Notice that the ideal starts in degree 3.

   (ii) Find the non-Lefschetz locus of A, i.e. the set of linear forms L for which $\times L : A_i \to A_{i+1}$ fails to have maximal rank for at least one i. [Hint 1: use without proof the fact that in this example it's enough to find the set of linear forms for which $A_1 \to A_2$ fails to be injective.] [Hint 2: R is an integral domain.] Surprising Moral: in this calculation I doesn't really play a role!!

(b) Let $I = \langle x^3, y^4 \rangle \subset \mathbb{C}[x,y] = R$.

(i) Find the Hilbert function of A.

(ii) Find the non-Lefschetz locus of A. [Hint: use without proof the fact that in this example it's enough to find the set of linear forms for which $A_2 \to A_3$ fails to be injective.]

Lecture 5. Geproci sets

1. If you know how to use a computer algebra program like Macaulay2 or CoCoA or Singular, verify that the standard construction (as described in the lecture) with $a = 4$ does indeed give three different geproci sets not lying on a quadric surface, namely $X \cup Y_1$, $X \cup Y_2$ and $X \cup Y_1 \cup Y_2$.

2. Let $Z \subset \mathbb{P}^3$ be a set of four points not all on a plane. Prove that Z is geproci.

3. Let $Z \subset \mathbb{P}^3$ be a set of three points.

   (a) Prove that Z is geproci if and only if the three points of Z all lie on a line.

   (b) Now assume that the three points of Z do not lie on a line, so by the first part Z is not geproci. Nevertheless, there is a locus of points P so that $\pi_P(Z)$ is a complete intersection. Find this locus.

4. Let $Z \subset \mathbb{P}^3$ be a set of 5 points such that no subset of 4 of them lies on a plane.

   (a) Prove that this implies that no three lie on a line.

   (b) Prove that Z is not geproci.

5. Let Z be a set of 6 general points in $\mathbb{P}^3$.

   (a) Prove that Z is not geproci. [Hint: prove that the general projection of Z does not lie on a conic. To do this, you can use without proof that it's enough to fine **one** projection whose image does not lie on a conic.]

   (b) It happens to be true that there is a locus of points from which the projection *does* lie on a conic. It turns out that this locus is a surface of degree 4 called the Weddle surface. (This is beyond the scope of these exercises.) Find some lines that lie on this surface. [Hint: there are at least 25 such lines, coming in two different kinds.]

   (c) Continuing from the previous exercise, it turns out that 6 general points in $\mathbb{P}^3$ lie on a unique twisted cubic curve. Prove that this curve also lies on the Weddle surface, i.e. prove that projecting from a general point on this curve sends the 6 points to 6 points on a conic in the target $\mathbb{P}^2$.

6. Let Q be the quadric surface defined by the equation $xy - zw = 0$. The corresponding coordinates will be understood to mean $[x, y, z, w]$.

   Notice that the line ℓ defined by $x = z = 0$ lies on Q as does the line m defined by $x = w = 0$, and that these lines meet at the point $[0, 1, 0, 0]$.

   Consider the following points on Q:

   $[0, 1, 0, 0]$ [0,1,1,0] [0,1,2,0] [0,1,3,0]

   $[0, 1, 0, 1]$

   $[0, 1, 0, 2]$

Extend this set to a grid on Q. (In the end you should have 12 points.)