Lecture 5: Geproci sets
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**Theorem.** (Bézout) Let \( C_1 \) and \( C_2 \) be plane curves defined by squarefree homogeneous polynomials \( f_1 \) and \( f_2 \), with no common component.

Assume \( \deg(f_1) = a \) and \( \deg(f_2) = b \).

Then \( C_1 \) and \( C_2 \) meet in at most \( ab \) points: \( |C_1 \cap C_2| \leq ab \).

Furthermore ...
If $C_1$ meets $C_2$ transversally at every intersection point then $|C_1 \cap C_2| = ab$. 
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$C_1$ ($a = 1$)

$C_2$ ($b = 2$)
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Definition. A set $Z$ of $ab$ distinct points in $\mathbb{P}^2$ is a complete intersection of type $(a, b)$ if there exist a curve $C_1$ of degree $a$ and a curve $C_2$ of degree $b$ such that $Z = C_1 \cap C_2$.

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Today we’ll look at a recently born area closely related to complete intersections.
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Let $H$ be a plane in $\mathbb{P}^3$ and let $P$ be a point not in $H$.

For us $P$ will usually be a general point.
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![Diagram](image)

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Let \( Q_1 \in \mathbb{P}^3 \), \( Q_1 \neq P \).

Let \( \lambda_{Q_1} \) be the line spanned by \( P \) and \( Q_1 \).
We also need the notion of a \textit{projection} $\pi_P$.

Let $H$ be a plane in $\mathbb{P}^3$ and let $P$ be a point not in $H$.

Define $\pi_P(Q_1)$ to be $H \cap \lambda_{Q_1}$. 
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Let $Q_1 \in \mathbb{P}^3$, $Q_1 \neq P$. Let $\lambda_{Q_1}$ be the line spanned by $P$ and $Q_1$. Define $\pi_P(Q_1)$ to be $H \cap \lambda_{Q_1}$.

We get the projection $\pi_P : \mathbb{P}^3 \setminus \{P\} \to H = \mathbb{P}^2$. 

Juan C. Migliore

Projective Planes and Beyond
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We get the projection $\pi_P : \mathbb{P}^3 \setminus \{P\} \to H = \mathbb{P}^2$. 
Let $d \geq 3$ be a positive integer and let $Z \subset P^3$ be a subset made of $d^2$ distinct points, with the following property: for a general projection $\pi: P^3 \to P^2$, the subset $\pi(Z) \subset P^2$ is the complete intersection of two plane curves of degree $d$.

Is it true that $Z$ itself is contained in a plane (and is the complete intersection of two curves of degree $d$)? If not, what is a counterexample?
Ancient History

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**Question.** (Francesco Polizzi, MathOverflow, June 8, 2011)

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*for a general projection $\pi : \mathbb{P}^3 \rightarrow \mathbb{P}^2$, the subset $\pi(Z) \subset \mathbb{P}^2$ is the complete intersection of two plane curves of degree $d$.*

Is it true that $Z$ itself is contained in a plane (and is the complete intersection of two curves of degree $d$)?

*If not, what is a counterexample?*
Comments:

1. It’s clear (as noted by Polizzi) that if $Z$ is itself already a complete intersection in some plane then $\pi_P(Z)$ is a complete intersection in $H$. We’ll call these trivial examples.

2. A non-degenerate set of 4 points is also an obvious example.

3. There is no reason to restrict to $d^2$ points. Better: does there exist a set $Z$ of $ab$ points in $P^3$ whose general projection is a complete intersection of type $(a,b)$? Without loss of generality assume $a \leq b$.

4. Trivial if $a = 1$. So assume $2 \leq a \leq b$. 

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   Without loss of generality assume $a \leq b$.

4. Trivial if $a = 1$. So assume $2 \leq a \leq b$. 
5. We’ll call such a set $Z$ a **geproci** set because its **General Projection** is a **Complete Intersection**.
5. We’ll call such a set $Z$ a \textit{geproci} set because its \textit{General Projection} is a \textit{Complete Intersection}.

6. For any values of $a, b$ there is a large class of slightly less obvious geproci sets $Z$, as noticed almost immediately in 2011 by Dmitri Panov and posted on MathOverflow:

On any smooth quadric surface $Q$ in $\mathbb{P}^3$, take $a$ lines, $L_1, \ldots, L_a$, from one ruling and $b$ lines, $M_1, \ldots, M_b$ from the other.
(Image from Wikipedia)
We have $|L_i \cap L_j| = 0$ and $|L_i \cap M_j| = 1$ for all $i, j$. 

Let $Z$ be the following intersection on $\mathbb{Q}$:

$$(L_1 \cup \cdots \cup L_a) \cap (M_1 \cup \cdots \cup M_b).$$

Then $|Z| = ab$.

The projection $\pi(Z)$ from a general point in $P_3$ is the complete intersection of the union of $a$ projected lines with a union of $b$ projected lines. So such a set is a nontrivial projective set.

We call such a set $Z$ a grid.
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New Version of our Question.

Do there exist nontrivial, non-grid geproci sets?

(For the rest of this question, “geproci” assumes nontrivial and non-grid.)
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- Do there exist nontrivial, non-grid geproci sets?

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- For which $a$ and $b$ do $(a, b)$-geproci sets exist?
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 ► *Can we classify the geproci sets somehow?*
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- Describe the geometry of geproci sets.
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- Can we classify the geproci sets somehow?

- Describe the geometry of geproci sets.

This problem went unnoticed for about a decade. I’ll talk about some recent answers.
More Recent History

At the 2018 conference "Lefschetz Properties and Jordan Type in Algebra, Geometry and Combinatorics" in Levico Terme, Italy, one workgroup working on something apparently unrelated stumbled on the first known non-trivial, non-grid example of a geproci set. We had no idea that Polizzi's question even existed!

This was followed by a paper by Luca Chiantini and JM (TAMS 2021), which included an appendix written by all participants of the workgroup.
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The main part of the paper introduced the problem, pointed out the grid example, and made connections to unexpected hypersurfaces. The term “geproci” was not yet introduced.
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The appendix gave all then-known geproci examples that arise in the same way that the original Levico observation arose.
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The appendix gave all then-known geproci examples that arise in the same way that the original Levico observation arose.

When the paper was uploaded to the arXiv, Polizzi pointed out to us his MathOverflow posting and the fact that grids were already known to have the general projection property.

The paper was modified to credit him with the question and to credit Panov with the grid observation.
Shortly afterwards, a group in Poland put out some papers introducing the term “geproci” and making further studies of the examples coming from the Levico work.

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Work continued very quickly and energetically, and we merged into a long-term project involving three of the Polish group, two Italians and two Americans.
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This group has now been working together extensively for about 2.5 years on the main questions mentioned above, and many related questions.

This is the POLITUS group.
Justyna Szpond
Giuseppe Favacchio
(This is why doing math is so much fun!!)
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Today we’ll talk about some of the results of this work.

All results mentioned from now on are from the POLITUS group.
The $D_4$ configuration

The smallest nontrivial, non-grid example of a geproci set is called the $D_4$ configuration.

It is a set of 12 points, not on a quadric surface (hence non-grid,) which nevertheless has a lot of collinearities.
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**Theorem.** Any nontrivial, non-grid geproci set of 12 points is projectively equivalent to $D_4$. 
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The smallest nontrivial, non-grid example of a geproci set is called the $D_4$ configuration.

It is a set of 12 points, not on a quadric surface (hence non-grid,) which nevertheless has a lot of collinearities.

Theorem. Any nontrivial, non-grid geproci set of 12 points is projectively equivalent to $D_4$.

Furthermore, its structure is representative of most known geproci sets.
What does $D_4$ look like?

The $D_4$ configuration is the union of a $(3, 3)$-grid, which lies on a unique quadric, and an additional set of three collinear points not on that quadric.
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Hence $D_4$ is not a grid.
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The $D_4$ configuration is the union of a $(3, 3)$-grid, which lies on a unique quadric, and an additional set of three collinear points not on that quadric.

Hence $D_4$ is not a grid.

There are several different ways to decompose $D_4$ as such a union!
Up to projective equivalence, the $D_4$ configuration consists of the points

- $[0, 0, 1, 0]$, $[0, 1, 1, 1]$, $[0, 1, 0, 1]$
- $[1, 0, 1, 1]$, $[0, 0, 1, 1]$, $[1, 0, 0, 0]$
- $[1, 0, 0, 1]$, $[0, 1, 0, 0]$, $[1, 1, 0, 1]$
- $[0, 0, 0, 1]$, $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1]$, $[1, 1, 1, 1]$
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[1, 0, 0, 1], [0, 1, 0, 0], [1, 1, 0, 1] \\
[0, 0, 0, 1], [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1], [1, 1, 1, 1]
\]

The top 9 points are a (3,3)-grid on a unique quadric $Q$. The last row is not on $Q$. 
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\begin{align*}
[0, 0, 1, 0], & \quad [0, 1, 1, 1], \quad [0, 1, 0, 1] \\
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[1, 0, 0, 1], & \quad [0, 1, 0, 0], \quad [1, 1, 0, 1] \\
[0, 0, 0, 1], & \quad [\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1], \quad [1, 1, 1, 1]
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Note: There are no four collinear points.
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The top 9 points are a (3,3)-grid on a unique quadric $Q$.

The last row is not on $Q$.

Note: There are no four collinear points.

Here’s one way to visualize it:
(Not visible: the back vertex point $[1, 1, 1, 1]$, the center point $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1]$ and the orthogonal lines through the point $[1, 1, 1, 1]$ along the three back edges.)
Theorem. (Classification of \((3, b)\)-geproci sets)

Let \(Z\) be a \((3, b)\)-geproci set \(Z\).

Then \(Z\) is necessarily a \((3, b)\)-grid (on a quadric surface), except for the case \(b = 4\), where \(Z\) can also be the \(D_4\) configuration (up to projective equivalence).
Theorem. (Classification of $(3, b)$-geproci sets)

Let $Z$ be a $(3, b)$-geproci set $Z$.

Then $Z$ is necessarily a $(3, b)$-grid (on a quadric surface), except for the case $b = 4$, where $Z$ can also be the $D_4$ configuration (up to projective equivalence).

Lemma. (Liaison trick)

Let $Z$ be $(a, b)$-geproci and assume that $Z$ contains a set of $a$ collinear points. Then the removal of these points is an $(a, b - 1)$-geproci set.

This also holds if we allow $a > b$. 
Example. Recall the $D_4$ configuration, which is $(3, 4)$-geproci:

$$
\begin{align*}
&[0, 0, 1, 0], [0, 1, 1, 1], [0, 1, 0, 1] \\
&[1, 0, 1, 1], [0, 0, 1, 1], [1, 0, 0, 0] \\
&[1, 0, 0, 1], [0, 1, 0, 0], [1, 1, 0, 1] \\
&[0, 0, 0, 1], \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right], [1, 1, 1, 1]
\end{align*}
$$

Obviously removing the last row leaves a $(3, 3)$-grid, which is geproci.
But also removing any other row leaves a $(3, 3)$-geproci set by the Lemma.
This set is also a grid by the CM result.
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1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \\
1 & 1 & 1 & 1 \\
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[0, 0, 0, 1], [1, 0, 0, 1], [0, 1, 0, 0], [1, 1, 0, 1] \\
[0, 0, 0, 1], [1/2, 1/2, 1/2, 1], [1, 1, 1, 1]
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Obviously removing the last row leaves a $(3, 3)$-grid, which is geproci.

But also removing any other row leaves a $(3, 3)$-geproci set by the Lemma. This set is also a grid by the CM result.
Theorem. (The Standard Construction)

Fix any integer \( a \geq 3 \).

Let \( u \) be a primitive \( a \)-th root of unity.
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Let $Q$ be the quadric surface in $\mathbb{P}^3$ defined by $xw - yz = 0$. 


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Let \( X \) be the following \( a^2 \) points; \( X \) forms a grid on \( Q \).

\[
\begin{align*}
[1, 1, 1, 1], & \quad [1, u, 1, u], & \ [1, u^2, 1, u^2], & \ldots & \ [1, u^{a-1}, 1, u^{a-1}] \\
[1, 1, u, u], & \quad [1, u, u, u^2], & \ [1, u^2, u, u^3], & \ldots & \ [1, u^{a-1}, u, 1] \\
[1, 1, u^2, u^2], & \quad [1, u, u^2, u^3], & \ [1, u^2, u^2, u^4], & \ldots & \ [1, u^{a-1}, u^2, u] \\
\vdots & \quad \vdots & \quad \vdots & \quad \vdots & \quad \vdots \\
[1, 1, u^{a-1}, u^{a-1}], & \quad [1, u, u^{a-1}, 1], & \ [1, u^2, u^{a-1}, u], & \ldots & \ [1, u^{a-1}, u^{a-1}, u^{a-2}]
\end{align*}
\]
Let

$\begin{align*}
Y_1 &= \{[1,0,0,-1], [1,0,0,-u], \ldots, [1,0,0,-u^{a-1}]\} \\
Y_2 &= \{[0,1,-1,0], [0,1,-u,0], \ldots, [0,1,-u^{a-1},0]\}.
\end{align*}$
Let
\[ Y_1 = \{ [1, 0, 0, -1], [1, 0, 0, -u], \ldots, [1, 0, 0, -u^{a-1}] \} \]
\[ Y_2 = \{ [0, 1, -1, 0], [0, 1, -u, 0], \ldots, [0, 1, -u^{a-1}, 0] \}. \]

The points \( Y_1 \) are collinear, the points \( Y_2 \) are collinear, and all lie off \( Q \).
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\[ Y_1 = \{ [1, 0, 0, -1], [1, 0, 0, -u], \ldots, [1, 0, 0, -u^{a-1}] \} \]

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The points \( Y_1 \) are collinear, the points \( Y_2 \) are collinear, and all lie off \( Q \). Then

(a) Both \( X \cup Y_1 \) and \( X \cup Y_2 \) are \((a, a + 1)\)-geproci.
Let
\[ Y_1 = \{ [1, 0, 0, -1], [1, 0, 0, -u], \ldots, [1, 0, 0, -u^{a-1}] \} \]
\[ Y_2 = \{ [0, 1, -1, 0], [0, 1, -u, 0], \ldots, [0, 1, -u^{a-1}, 0] \}. \]

The points \( Y_1 \) are collinear, the points \( Y_2 \) are collinear, and all lie off \( Q \). Then

(a) Both \( X \cup Y_1 \) and \( X \cup Y_2 \) are \((a, a + 1)\)-geproci.

(b) If \( a \) is even then in addition, \( X \cup Y_1 \cup Y_2 \) is \((a, a + 2)\)-geproci.
Let

\[ Y_1 = \{[1, 0, 0, -1], [1, 0, 0, -u], \ldots, [1, 0, 0, -u^{a-1}]\} \]

\[ Y_2 = \{[0, 1, -1, 0], [0, 1, -u, 0], \ldots, [0, 1, -u^{a-1}, 0]\} . \]

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Since these geproci sets do not lie on a quadric, they are not grids.
Note the similarity with $D_4$ ($(3, 3)$-grid plus 3 collinear points). In fact $D_4$ comes from the standard construction ($a = 3$) (up to projective equivalence).

Corollary. (Numerical classification of geproci sets)

Fix integers $a, b$ with $1 \leq a \leq b$.

$\begin{align*}
\Rightarrow & \text{ If } a = 1 \text{ or } a = 2 \text{ then there are no non-trivial, non-grid geproci sets.} \\
\Rightarrow & \text{ If } a = 3 \text{ then the only non-trivial, non-grid geproci set is } D_4 \text{ (up to projective equivalence), which is } (3, 4)\text{-geproci.} \\
\Rightarrow & \text{ If } 4 \leq a \leq b \text{ then there is a non-trivial, non-grid } (a, b)\text{-geproci set.}
\end{align*}$

Idea of proof: It is a consequence of the liaison trick. (Example coming.)
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It is a consequence of the liaison trick. (Example coming.)
Example. How would we construct a $(4, 8)$-geproci set?

Step 1: Use the standard construction to construct an $(8, 9)$-geproci set. This consists of an $(8, 8)$-grid, $X$, on a quadric $Q$ plus a set $Y_1$ of 8 collinear points not on $Q$.

Step 2: Use the liaison trick to remove a set of 8 collinear points from $X$, giving an $(8, 9 - 8) = (8, 8)$-geproci set.
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Step 3: Take four more sets of 8 collinear points from the same ruling of the grid, to get the \((8, 8 - 4) = (8, 4) = (4, 8)\)-geproci set.
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Thus we have a \((4, 8)\) non-grid geproci set.
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- \(a\) lines with \(b\) collinear points each.

But in fact there exist nontrivial, non-grid geproci sets that are not half-grids.

But only a very small number are known, and they have large subsets that are half-grids.
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5. And there are no known geproci sets in \(\mathbb{P}^n\) for \(n \geq 4\).
Luca Chiantini, Łucja Farnik, Giuseppe Favacchio, Brian Harbourne, Juan Migliore, Tomasz Szemberg, Justyna Szpond

*Configurations of points in projective space and their projections*