Projective Planes and Beyond

Thematic Program on Rationality and Hyperbolicity

Undergraduate Workshop

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June 12-16, 2023

Slides available by emailing migliore.1@nd.edu
or from the conference website.

Lecture 5: Geproci sets

Introduction

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Theorem. (Bézout) *Let C_1 and C_2 be plane curves defined by squarefree homogeneous polynomials f_1 and f_2, with no common component.*

Assume $\deg(f_1) = a$ *and* $\deg(f_2) = b$.

Then C_1 and C_2 meet in at most ab points: $|C_1 \cap C_2| \leq ab$.

Furthermore ...

- *If C_1 meets C_2 transversally at every intersection point then $|C_1 \cap C_2| = ab$.*

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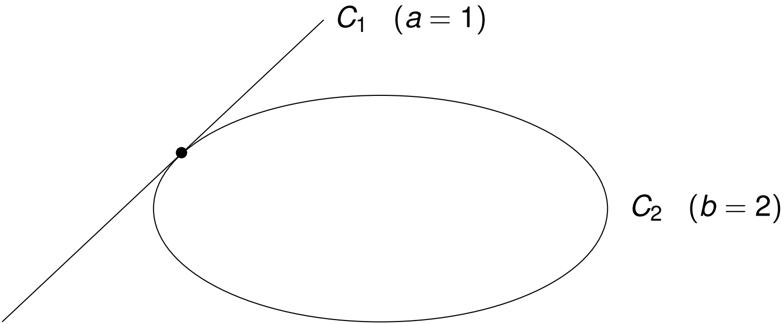

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Definition. A set Z of ab distinct points in $\mathbb{P}^2$ is a complete intersection of type (a, b) if there exist a curve C_1 of degree a and a curve C_2 of degree b such that $Z = C_1 \cap C_2$.

We've mentioned complete intersections a few times this week.

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Today we'll look at a recently born area closely related to complete intersections.

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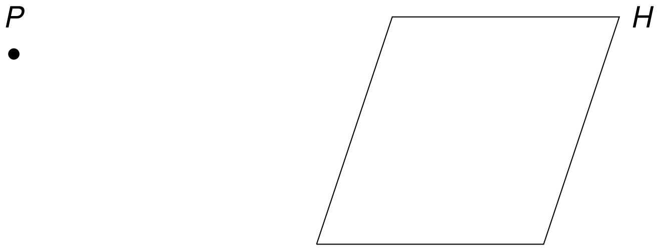

For us P will usually be a general point.

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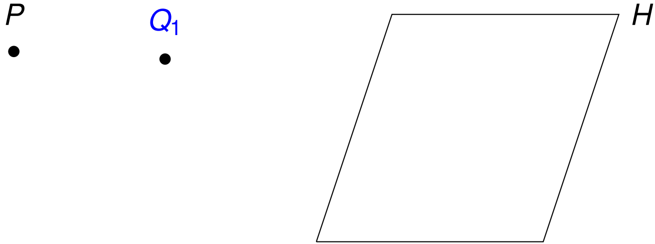

Let $Q_1 \in \mathbb{P}^3$, $Q_1 \neq P$.

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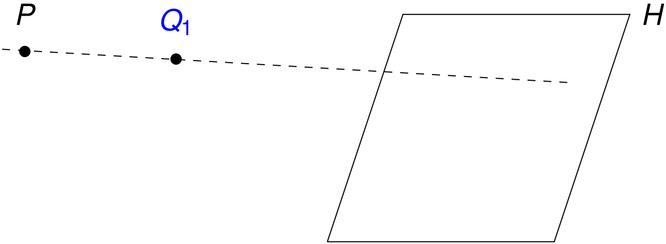

Let λ_{Q_1} be the line spanned by P and Q_1.

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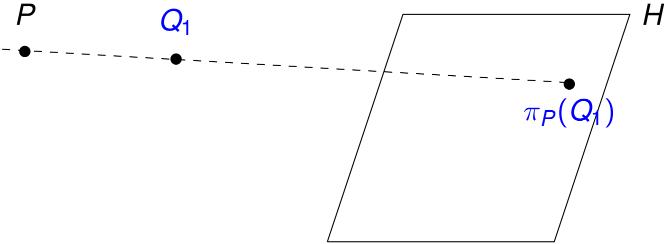

Define $\pi_P(Q_1)$ to be $H \cap \lambda_{Q_1}$.

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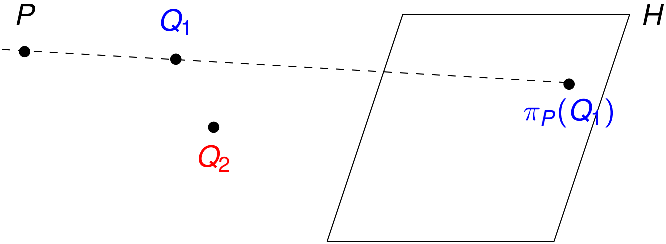

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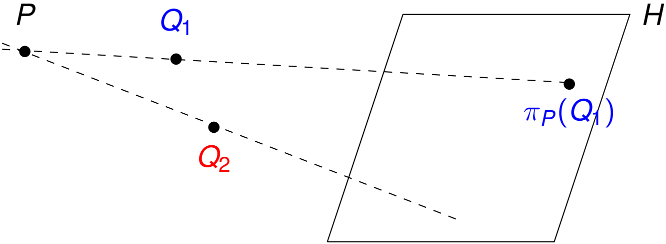

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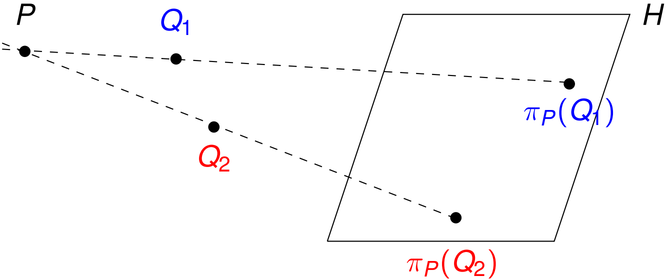

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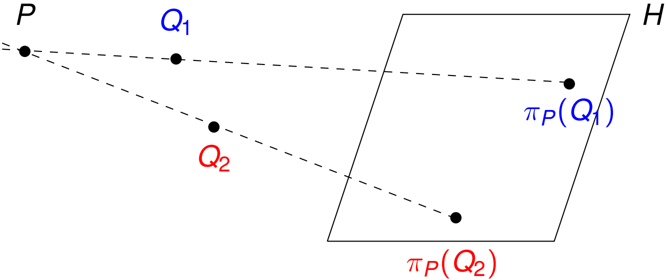

We get the projection $\pi_P : \mathbb{P}^3 \backslash \{P\} \to H = \mathbb{P}^2$.

Ancient History

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Question. (Francesco Polizzi, MathOverflow, June 8, 2011)

Let $d \geq 3$ be a positive integer and let $Z \subset \mathbb{P}^3$ be a subset made of d^2 distinct points, with the following property:

> *for a **general** projection $\pi : \mathbb{P}^3 \to \mathbb{P}^2$, the subset $\pi(Z) \subset \mathbb{P}^2$ is the complete intersection of two plane curves of degree d.*

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Is it true that Z itself is contained in a plane (and is the complete intersection of two curves of degree d)?

If not, what is a counterexample?

Comments:

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3. There is no reason to restrict to d^2 points. Better: does there exist a set Z of ab points in $\mathbb{P}^3$ whose general projection is a complete intersection of type (a, b)?

   Without loss of generality assume $a \leq b$.

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4. Trivial if $a = 1$. So assume $2 \leq a \leq b$.

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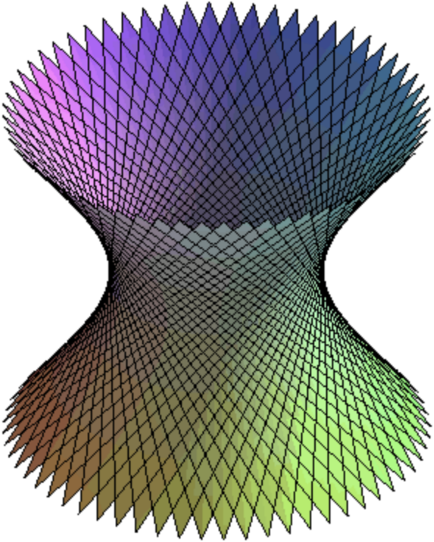
6. For any values of a, b there is a large class of slightly less obvious geproci sets Z, as noticed almost immediately in 2011 by Dmitri Panov and posted on MathOverflow:

   On any smooth quadric surface Q in $\mathbb{P}^3$, take a lines, $L_1, \dots .L_a$, from one ruling and b lines, $M_1, \dots, M_b$ from the other.



(Image from Wikipedia)

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Let Z be the following intersection on Q:

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Then $|Z| = ab$.

The projection $\pi(Z)$ from a general point in $\mathbb{P}^3$ is the complete intersection of the union of a projected lines with a union of b projected lines. So such a set is a nontrivial geproci set.

We call such a set Z a grid.

New Version of our Question.

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This problem went unnoticed for about a decade. I'll talk about some recent answers.

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At the 2018 conference

"Lefschetz Properties and Jordan Type in Algebra, Geometry and Combinatorics"

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This was followed by a paper by Luca Chiantini and JM (TAMS 2021), which included an appendix written by all participants of the workgroup.

The main part of the paper introduced the problem, pointed out the grid example, and made connections to unexpected hypersurfaces. The term “geproci” was not yet introduced.

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When the paper was uploaded to the arXiv, Polizzi pointed out to us his MathOverflow posting and the fact that grids were already known to have the general projection property.

The paper was modified to credit him with the question and to credit Panov with the grid observation.

Shortly afterwards, a group in Poland put out some papers introducing the term “geproci” and making further studies of the examples coming from the Levico work.

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This group has now been working together extensively for about 2.5 years on the main questions mentioned above, and many related questions.

This is the POLITUS group.

POLAND ITALY US

Łucja Farnik

Tomasz Szemberg

Justyna Szpond

Luca Chiantini

Giuseppe Favacchio

Brian Harbourne

me

(This is why doing math is so much fun!!)

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Today we'll talk about some of the results of this work.

All results mentioned from now on are from the POLITUS group.

The D_4 configuration

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Theorem. *Any nontrivial, non-grid geproci set of 12 points is projectively equivalent to D_4.*

Furthermore, its structure is representative of most known geproci sets.

What does D_4 look like?

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There are several different ways to decompose D_4 as such a union!

Up to projective equivalence, the D_4 configuration consists of the points

$$\begin{gathered}[0,0,1,0],[0,1,1,1],[0,1,0,1]\\ [1,0,1,1],[0,0,1,1],[1,0,0,0]\\ [1,0,0,1],[0,1,0,0],[1,1,0,1]\end{gathered}$$

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$$[0,0,0,1], [\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, 1], [1,1,1,1]$$

The top 9 points are a (3,3)-grid on a unique quadric $\mathcal{Q}$.

The last row is not on $\mathcal{Q}$.

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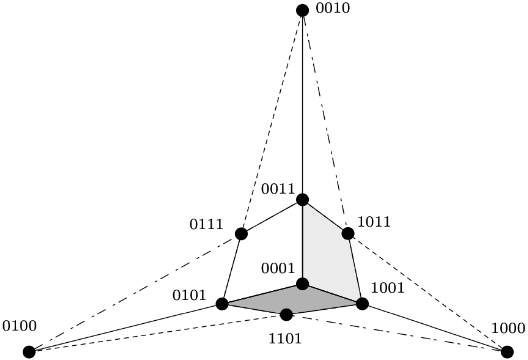
Here's one way to visualize it:



(Not visible: the back vertex point [1, 1, 1, 1], the center point $[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1]$ and the orthogonal lines through the point [1, 1, 1, 1] along the three back edges.)

Some Main Results

Theorem. (Classification of $(3, b)$-geproci sets)

Let Z be a $(3, b)$-geproci set Z.

Then Z is necessarily a $(3, b)$-grid (on a quadric surface), except for the case $b = 4$, where Z can also be the D_4 configuration (up to projective equivalence).

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Lemma. (Liaison trick)
Let Z be (a, b)-geproci and assume that Z contains a set of a collinear points. Then the removal of these points is an $(a, b - 1)$-geproci set.

This also holds if we allow $a > b$.

Example. Recall the D_4 configuration, which is $(3,4)$-geproci:

$$[0,0,1,0],[0,1,1,1],[0,1,0,1]$$
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Obviously removing the last row leaves a $(3,3)$-grid, which is geproci.

But also removing any other row leaves a $(3,3)$-geproci set by the Lemma. This set is also a grid by the CM result.

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Let X be the following a^2 points; X forms a grid on $\mathbb{Q}$.

$$
\begin{array}{lllll}
[1,1,1,1], & [1,u,1,u], & [1,u^2,1,u^2], & \dots & [1,u^{a-1},1,u^{a-1}] \\
[1,1,u,u], & [1,u,u,u^2], & [1,u^2,u,u^3], & \dots & [1,u^{a-1},u,1] \\
[1,1,u^2,u^2], & [1,u,u^2,u^3], & [1,u^2,u^2,u^4], & \dots & [1,u^{a-1},u^2,u] \\
\vdots & \vdots & \vdots & & \vdots \\
[1,1,u^{a-1},u^{a-1}], & [1,u,u^{a-1},1], & [1,u^2,u^{a-1},u], & \dots & [1,u^{a-1},u^{a-1},u^{a-2}]
\end{array}
$$

Let

- $Y_1 = \{[1,0,0,-1],[1,0,0,-u],\dots,[1,0,0,-u^{a-1}]\}$
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(a) *Both $X \cup Y_1$ and $X \cup Y_2$ are $(a, a+1)$-geproci.*

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The points Y_1 are collinear, the points Y_2 are collinear, and all lie off $\mathcal{Q}$. Then

(a) *Both $X \cup Y_1$ and $X \cup Y_2$ are $(a, a+1)$-geproci.*

(b) *If a is even then in addition, $X \cup Y_1 \cup Y_2$ is $(a, a+2)$-geproci.*

Let

- $Y_1 = \{[1,0,0,-1],[1,0,0,-u],\dots,[1,0,0,-u^{a-1}]\}$
- $Y_2 = \{[0,1,-1,0],[0,1,-u,0],\dots,[0,1,-u^{a-1},0]\}$.

The points Y_1 are collinear, the points Y_2 are collinear, and all lie off $\mathcal{Q}$. Then

(a) *Both $X \cup Y_1$ and $X \cup Y_2$ are $(a, a+1)$-geproci.*

(b) *If a is even then in addition, $X \cup Y_1 \cup Y_2$ is $(a, a+2)$-geproci.*

Since these geproci sets do not lie on a quadric, they are not grids.

Note the similarity with D_4 ($(3,3)$-grid plus 3 collinear points). In fact D_4 comes from the standard construction ($a = 3$) (up to projective equivalence).

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Fix integers a, b with $1 \leq a \leq b$.

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Idea of proof:

It is a consequence of the liaison trick. (Example coming.) □

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This consists of an $(8, 8)$-grid, X, on a quadric $\mathcal{Q}$ plus a set Y_1 of 8 collinear points not on $\mathcal{Q}$.

Step 2: Use the liaison trick to remove a set of 8 collinear points from X, giving an $(8, 9 - 8) = (8, 8)$-geproci set.

Step 3: Take four more sets of 8 collinear points from the same ruling of the grid, to get the $(8, 8-4) = (8, 4) = (4, 8)$-geproci set.

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Thus we have a $(4, 8)$ non-grid geproci set.

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   - b lines with a collinear points each, or
   - a lines with b collinear points each.

   But in fact there exist nontrivial, non-grid geproci sets that are not half-grids.

   But only a very small number are known, and they have large subsets that are half-grids.

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5. And there are no known geproci sets in $\mathbb{P}^n$ for $n \geq 4$.

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Configurations of points in projective space and their projections

arXiv:2209.04820.