RATIONALITY AND HYPERBOLICITY SUMMER SCHOOL: RATIONALITY OF THREEFOLDS OVER NON-CLOSED FIELDS EXERCISES

SARAH FREI

Lecture 2: The intermediate Jacobian obstruction

Some background on abelian varieties:[1] Let V be a g-dimensional complex vector space, and $\Lambda \subset V$ a lattice (which means a discrete subgroup of rank $2g$, i.e. $\Lambda \cong \mathbb{Z}^{2g}$, such that $\operatorname{Span}_{\mathbb{R}} \Lambda = V$).

A **polarization** on V/Λ is a non-degenerate, skew-symmetric bilinear form $q\colon \Lambda \times \Lambda \to \mathbb{Z}$ such that

(1) $q_{\mathbb{R}}\colon V \times V \to \mathbb{R}$ satisfies $q_{\mathbb{R}}(iv, iw) = q_{\mathbb{R}}(v, w)$, and
(2) the Hermitian form $H(v, w) := q_{\mathbb{R}}(iv, w) + iq_{\mathbb{R}}(v, w)$ is positive definite.

The polarization is **principal** if q is unimodular (which means that $\Lambda^\vee := \operatorname{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z}) \cong \Lambda$).

A complex abelian variety A of dimension g is a complex torus V/Λ equipped with a polarization. There is an isomorphism $H_1(A, \mathbb{Z}) \cong \Lambda$, where $V = H^0(A, \Omega^1_A)^\vee$, and the inclusion $H_1(A, \mathbb{Z}) \hookrightarrow H^0(A, \Omega^1_A)^\vee$ is given by

$$\gamma \mapsto \left(\omega \mapsto \int_\gamma \omega\right).$$

There is an association between polarizations as defined above and ample line bundles, so that q corresponds to a divisor class θ, and q is principal if and only if $H^0(A, \mathcal{O}_A(\theta)) = 1$. In this case, the class θ gives a well defined divisor, Θ, which is well-defined up to translation on A.

There is a natural homomorphism $\Lambda \to \Lambda^\vee$, and this induces a map $\lambda\colon A \to A^\vee := V^\vee/\Lambda^\vee$. We call $A^\vee$ the dual abelian variety to A.

A **principally polarized abelian variety** is an abelian variety equipped with a principal polarization.

Exercise 1. Show that a complex abelian variety is principally polarized if and only if it is isomorphic to its dual. *(This follows directly from the definition, so this exercise is just to check that the definitions make sense.)*

Date: June 20, 2023.

[1]This will be less helpful for today's exercises and more-so for Lecture 3; it's included here if you want to think more about abelian varieties and the role they played in Lecture 2.

Throughout, we will write $h^{p,q}(X) := \dim H^{p,q}(X)$; these dimensions are called the Hodge numbers of X. They are collected in a Hodge diamond–for example, for a surface:

$$\begin{array}{ccccc} & & h^{2,2} & & \\ & h^{2,1} & & h^{1,2} & \\ h^{2,0} & & h^{1,1} & & h^{0,1} \\ & h^{1,0} & & h^{0,1} & \\ & & h^{0,0} & & \end{array}$$

Exercise 2 (Warm-up with Hodge numbers)**.**

(1) Make sure you understand why, for example for a surface, all other $h^{p,q} = 0$ outside of those in the diamond.
(2) Let X be a smooth projective complex surface. Write $\chi(X, \mathcal{O}_X)$ in terms of Hodge numbers.
(3) Write down the Hodge diamond for $\mathbb{P}^2$.
(4) Show that the Hodge diamond is symmetric across the vertical center axis, and also has 180° rotational symmetry. *Hint: Serre duality.*

Exercise 3. Let X be a smooth complex cubic threefold. Show that $J^2(X) \cong J(\mathrm{Bl}_\ell X)$ for ℓ a line in X.

Exercise 4. For a slight variation on Exercise 1.7 (from day 1), let $X \subset \mathbb{P}^4$ be a one nodal cubic threefold, i.e. the singular locus is one ordinary double point; see [Huy23, Sections 1.5.4, 5.5.1] for more on this setting. By a node $p \in X$, we mean that the exceptional divisor $E_p \subset \mathrm{Bl}_p X$ is a smooth quadric surface when considered as a subvariety of the exceptional divisor $E \cong \mathbb{P}^3 \subset \mathrm{Bl}_p \mathbb{P}^4$.

(1) First, show that X is rational.
(2) After a linear change of coordinates, we may assume the singular point of X is $p = [0:0:0:0:1] \in X \subset \mathbb{P}^4$. Show that X can be written as $X = V(F + x_4 G)$ for $F \in H^0(\mathbb{P}^3, \mathcal{O}(3))$ and $G \in H^0(\mathbb{P}^3, \mathcal{O}(2))$, where $\mathbb{P}^3 = V(x_4)$.
(3) Show that when you resolve the birational isomorphism from part (1) as

$$\begin{array}{ccc} & \mathrm{Bl}_p X & \\ \swarrow & & \searrow \\ X & \dashrightarrow & \mathbb{P}^3 \end{array}$$

there is an isomorphism $\mathrm{Bl}_p X \cong \mathrm{Bl}_C \mathbb{P}^3$, where $C = V(F) \cap F(G)$.
(4) Conclude that $J(\mathrm{Bl}_p X) \cong J(C)$. This can be taken to be the definition of $J(X)$. Show also that $\dim J(X) = 4$, which differs from the case of a smooth cubic threefold.

Exercise 5. Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold. Show that $h^{1,2}(X) = h^{2,1}(X) = \dim H^2(X, \Omega^1_X) = 5$. *If you aren't familiar with computing Hodge numbers, an approach is outlined below. The method relies on cohomology long exact sequences from short exact sequences of bundles on X and $\mathbb{P}^4$.*

The proof proceeds via two main equalities:

(A) $h^2(X, \Omega^1_X) = h^3(X, \mathcal{O}_X(-3))$, and
(B) $h^3(X, \mathcal{O}_X(-3)) = h^0(X, \mathcal{O}_X(1))$.

(1) Use (A) and (B) along with the divisor short exact sequence
$$0 \to \mathcal{O}_{\mathbb{P}^4}(-3) \to \mathcal{O}_{\mathbb{P}^4} \to \mathcal{O}_X \to 0$$
to show the desired equality, that $h^{1,2}(X) = 5$.
(2) Use adjunction to show that $\omega_X \cong \mathcal{O}_X(-2)$.
(3) Show (B) using Serre duality.
(4) From the cotangent sequence
$$0 \to \mathcal{O}_X(-3) \to \Omega_{\mathbb{P}^4}|_X \to \Omega_X \to 0,$$
where $I_X/I_X^2 \cong \mathcal{O}_X(-3)$, write out the long exact sequence on cohomology. Show that (A) holds if $h^2(X, \Omega_{\mathbb{P}^2}|_X) = h^3(X, \Omega_{\mathbb{P}^4}|_X) = 0$.
(5) Finally, show the vanishing in part (4) using the long exact sequence on cohomology from the Euler sequence:
$$0 \to \Omega_{\mathbb{P}^4}|_X \to \mathcal{O}_X(-1)^{\oplus 5} \to \mathcal{O}_X \to 0.$$
To show the vanishing of the cohomology groups of $\mathcal{O}_X(-1)$, use the divisor short exact sequence from (1) along with Serre duality.

References

[ABB14] Asher Auel, Marcello Bernardara, and Michele Bolognesi, *Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems*, J. Math. Pures Appl. (9) **102** (2014), no. 1, 249–291, DOI 10.1016/j.matpur.2013.11.009 (English, with English and French summaries). MR3212256 ↑

[GH94] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original. MR1288523 ↑

[Har92] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1992. A first course. MR1182558 ↑

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. MR0463157 ↑

[Huy23] Daniel Huybrechts, *The geometry of cubic hypersurfaces*, Cambridge Studies in Advanced Mathematics, vol. 206, Cambridge University Press, Cambridge, 2023. MR4589520 ↑4

[Poo17] Bjorn Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. MR3729254 ↑

Email address: `sarah.frei@dartmouth.edu`
URL: `http://math.dartmouth.edu/~sfrei`