RATIONALITY AND HYPERBOLICITY SUMMER SCHOOL: RATIONALITY OF THREEFOLDS OVER NON-CLOSED FIELDS EXERCISES

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1. Lecture 4: Examples and generalizing to non-closed fields

Exercise 1. Let X be a smooth complex cubic threefold. Show that $J^2(X) \cong \mathrm{Prym}_{\tilde{\Delta}/\Delta}$ for Δ a plane quintic curve. *Hint: recall that we saw this curve back in Exercise 8 from Lecture 1.*

Exercise 2. Let $Q \subset \mathbb{P}^5$ be a smooth quadric 4-fold.

(1) Show that Q contains 2-planes.
Here's the idea: For any point $x \in Q$, $T_xQ \cap Q$ is a quadric of dimension 3. Check that it is singular, and it is the cone over a smooth quadric surface Q' (See [Har92, Lecture 22] *for a nice introduction to quadrics). Now use lines on Q' to produce 2-planes on Q.*

(2) Let $F := F_2(Q)$, the Fano variety of 2-planes on Q. Show that $\dim F = 3$.
Use the incidence correspondence $\Psi := \{(x, \Lambda) : x \in \Lambda \subset Q\} \subset Q \times F$. Show that for $x \in Q$, the fiber of $\Psi \to Q$ over x is 1-dimensional. Show that for $\Lambda \in F$, the fiber of $\Psi \to F$ over Λ is 2-dimensional. (cf. Exercise 3 from Lecture 1)

(3) What can you say about 2-planes if Q is not smooth, but is rather a cone over a smooth quadric 3-fold?

(4) Show that if Q is smooth, $F_2(Q)$ has two connected components, and if Q is a cone over a smooth quadric 3-fold, $F_2(Q)$ has only one connected component.

Exercise 3. Let X be a smooth threefold complete intersection of two quadrics. This exercise outlines the proof of Reid's result that $J^2(X) \cong J(C)$.

Let $k = \mathbb{C}$ and fix a line $s \subset X$. Since X is the base locus of $\mathcal{Q} \to \mathbb{P}^1$, this gives a line in Q_λ for all $\lambda \in \mathbb{P}^1$. Let $\tilde{C}_s := \{(\lambda, \Lambda) : s \subset \Lambda \subset Q_\lambda\} \subset \mathcal{F}_2(\mathcal{Q}/\mathbb{P}^1)$, so that $\tilde{C}_s$ parametrizes the 2-planes in the fibers of $\mathcal{Q} \to \mathbb{P}^1$ which contain the line s.

(1) Show that $\tilde{C}_s$ is a smooth curve and that $p|_{\tilde{C}_s} : \tilde{C}_s \to C$ is an isomorphism.

(2) Show that there is a morphism $r : \tilde{C}_s \to F_1(X)$. *Hint: Given a 2-plane $\Lambda \subset Q_\lambda$, consider $\Lambda \cap X$.*
Let $r' : C \to F_1(X)$ be r precomposed with the isomorphism $C \xrightarrow{\sim} \tilde{C}_s$, and $\Gamma_{r'} \in \mathrm{CH}^2(C \times F_1(X))$ the cycle class of the graph of r'.

(3) Show that $\Gamma_{r'}$ induces a map $J(C) \to \mathrm{Alb}(F_1(X))$.
Next, let $T := \{(s, x) : x \in s\} \subset F_1(X) \times X$, and consider $T \in \mathrm{CH}^2(F_1(X) \times X)$.

(4) Show that T induces a map $\mathrm{Alb}(F_1(X)) \to J^2(X)$.

Reid shows that the two morphisms in (3) and (4) are isomorphisms.

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Exercise 4. Let C be a geometrically rational curve over a field k. Show that C is rational if and only if $C(k) \neq \emptyset$.

The Lang-Nishimura Theorem is overkill here; you can show that C being rational implies $C(k) \neq \emptyset$ without it.

The following exercises are to familiarize yourself with torsors and the Weil-Châtelet group.

Let G be a smooth algebraic group scheme over a field k. A G-**torsor over** k (or a **torsor under** G or a **principal homogeneous space for** G) is a k-variety X equipped with a right action of G such that $X_{\bar{k}}$ equipped with the right action of $G_{\bar{k}}$ is isomorphic to $G_{\bar{k}}$ (equipped with the right action of translation).

This is equivalent to saying that there is a morphism $\mu\colon G \times X \to X$ (giving the action of G on X) for which $\mu(\bar{k})\colon G(\bar{k}) \times X(\bar{k}) \to X(\bar{k})$ is a simple transitive action of $G(\bar{k})$ on $X(\bar{k})$.

Exercise 5. Check that Pic^d_C satisfies the definition of being a Pic^0_C-torsor.

Exercise 6 (From [Poo17, Examples 5.5.3 and 5.12.8])**.** Let $T := V(x^2 + 2y^2 - 1) \subset \mathbb{A}^2_{\mathbb{Q}}$ and $X := V(x^2 + 2y^2 + 3) \subset \mathbb{A}^2_{\mathbb{Q}}$.

(1) Show that T is a group scheme with multiplication given by

$$\begin{aligned} m\colon T \times T &\to T \\ ((x_1, y_1), (x_2, y_2)) &\mapsto (x_1x_2 - 2y_1y_2, x_1y_2 + y_1x_2). \end{aligned}$$

(In fact $T_{\bar{\mathbb{Q}}} \cong \mathbb{G}_{m,\bar{\mathbb{Q}}}$.)

(2) Show that X is a T-torsor over $\mathbb{Q}$.
(In fact, it is a non-trivial torsor since $X(\mathbb{Q}) = \emptyset$.)

Exercise 7. Let C be a smooth projective genus one curve over k (and note that $C(k)$ may be empty). Show that C is a torsor under the elliptic curve Pic^0_C.

The collection of all G-torsors up to isomorphism is parametrized by the cohomology set $H^1(k, G(\bar{k}))$. When G is commutative, this cohomology set is an *abelian group.*

For an abelian variety A, the **Weil-Châtelet group**

$$\mathrm{WC}(A) := \{\text{torsors under } A\}/\text{isomorphism}$$

is an abelian group.

Exercise 8. The group operation on the Weil-Châtelet group of an abelian variety A can be described as follows. Let T_1 and T_2 be A-torsors, and $T_1 \times^A T_2$ the quotient of $T_1 \times T_2$ by the A-action where, for $a \in A$, $a \cdot (t_1, t_2) := (a + t_1, [-1]a + t_2)$ (we write the action additively since A is abelian!). Here, $[-1]\colon A \to A$ is the standard involution on A (i.e. it is the inverse morphism for the group structure on A).

(1) Show that the diagonal action of A on $T_1 \times T_2$ descends to an action on $T_1 \times^A T_2$.
(2) Let $[T_1] + [T_2] := [T_1 \times^A T_2]$. Show that this operation gives a group law on $\mathrm{WC}(A)$, where the inverse of a torsor T is T with the action $a \cdot t := [-1]a + t$.

For an A-torsor X, we will write $[X]$ for its class in $\mathrm{WC}(A)$. We list here some of the nice properties of $\mathrm{WC}(\mathrm{Pic}^0_C)$ for C a smooth projective curve:

(1) $[\mathrm{Pic}^0_C] = 0$.
(2) $[T] = 0$ if and only if $T(k) \neq \emptyset$ if and only if $T \cong \mathrm{Pic}^0_C$.
(3) For all $d \in \mathbb{Z}$, $[\mathrm{Pic}^d_C] = d[\mathrm{Pic}^1_C]$.

Exercise 9. Use the structure of Pic_C and the description of the group operation in Exercise 8 to prove the third property of the Weil-Châtelet group of Pic^0_C: that for all $d \in \mathbb{Z}$, $[\mathrm{Pic}^d_C] = d[\mathrm{Pic}^1_C]$.

Exercise 10. Let C be a smooth projective curve over k of genus $g \geq 2$, and let $m = 2g-2$. Show that for all $t \in \mathbb{Z}$, $[\mathrm{Pic}^{tm}_C] = [\mathrm{Pic}^0_C] = 0$.

This shows that the subgroup $\langle \mathrm{Pic}^1_C \rangle \leqslant \mathrm{WC}(\mathrm{Pic}^0_C)$ is a finite cyclic group, and the order of this subgroup is called the period of C.

References

[Har92] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1992. A first course. MR1182558 ↑1

[Poo17] Bjorn Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. MR3729254 ↑6