RATIONALITY AND HYPERBOLICITY SUMMER SCHOOL: RATIONALITY OF THREEFOLDS OVER NON-CLOSED FIELDS EXERCISES

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1. Lecture 1: Measures of rationality and examples

Exercise 1.1. We showed the following theorem in lecture:

Theorem 1. *Let X and Y be smooth projective varieties with* $\dim X = \dim Y$*, and $X \dashrightarrow Y$ a dominant rational map. Then*

$$h^0(X, (\Omega_X^q)^{\otimes m}) \geq h^0(Y, (\Omega_Y^q)^{\otimes m}),$$

for all $q, m \geq 0$.

Check that the same result holds over any algebraically closed field of positive characteristic if you add the assumption that the birational map is separable. *(This is more of a remark than an exercise, but use it as an opportunity to reflect on the proof of the theorem.)*

Exercise 1.2. Show that a rational map $X \dashrightarrow Y$ between smooth projective varieties can be represented by a morphism $f\colon U \to Y$ with $\operatorname{codim}_X X \setminus U \geq 2$.

Exercise 1.3. Rather than relying on knowledge of del Pezzo surfaces, let's show directly that a smooth complete intersection of two quadric threefolds in $\mathbb{P}^4$ contains lines.

(1) Let $Q \subset \mathbb{P}^4$ be a smooth quadric threefold. Let $F_1(Q) \subset \mathrm{Gr}(2,5)$[1] be the Fano variety of lines in Q. Show that $\dim F_1(Q) = 3$.

Hint: Use the incidence correspondence $\Psi := \{(x, L) : x \in L \subset Q\} \subset Q \times F_1(Q)$. Show that for $x \in Q$, the fiber of $\Psi \to Q$ over x is 1-dimensional. Show that for $L \in F_1(Q)$, the fiber of $\Psi \to F_1(Q)$ over L is 1-dimensional.

(2) Show that for quadric threefolds $Q_0, Q_1 \subset \mathbb{P}^4$ which intersect transversally, we have $\dim F_1(Q_0) \cap F_1(Q_1) = 0$, so that there are finitely many lines contained in the intersection $Q_0 \cap Q_1$.

Indeed, the degree of each $F_1(Q_i) \subset \mathrm{Gr}(2,5)$ is 4, so that we recover the 16 lines mentioned above.

Exercise 1.4. Recall that we showed that a smooth cubic hypersurface of dimension $2m$ containing two disjoint m-planes is always rational. Why does this construction not work if the two m-planes are distinct but not disjoint?

Exercise 1.5. Write down an explicit equation of a smooth cubic hypersurface of dimension $2m$ containing two disjoint m-planes.

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[1]Note that we use affine dimensions for the Grassmannian throughout, so that $\mathrm{Gr}(2,5)$ parametrizes lines in $\mathbb{P}^4$.

Exercise 1.6. Let S be a smooth cubic surface. We just showed that S is unirational, and we showed that any unirational surface is rational, hence proving the rationality of smooth cubic surfaces.

Show that S contains two disjoint lines, giving another proof of rationality. *Hint: Use the fact that S is the blow-up of $\mathbb{P}^2$ at 6 points in general position, see e.g.* [Har77, Section V.4].

Exercise 1.7. Let X be a cubic threefold containing a plane P. Pick coordinates on $\mathbb{P}^4$ so that $P = V(x_0, x_1)$. Then we can write $X = V(x_0q_0 + x_1q_1)$ for quadrics $q_0, q_1 \in k[x_0, x_1, x_2, x_3, x_4]$.

(1) Show that the singular locus of X contains $P \cap V(q_0) \cap V(q_1)$, which is the intersection of two conics in P.
(2) Suppose we are in the general case, so this is exactly the singular locus (four nodes contained in P). Show that projection from a singular point gives a birational isomorphism $X \overset{\sim}{\dashrightarrow} \mathbb{P}^3$.
(3) Show that X contains a line which isn't contained in P. You might be temped to use this line for a rationality construction (e.g. writing down a rational map $L \times P \dashrightarrow X$). Why doesn't this work?
(4) Show that X contains a line disjoint from P, and use this line to give a different rationality construction.

Exercise 1.8. Let $X \subset \mathbb{P}^4$ be a smooth cubic threefold, and $\pi\colon X' := \mathrm{Bl}_\ell X \to \mathbb{P}^2$ the associated conic bundle. We will show in this exercise that the discriminant divisor, which in this case is a plane curve, has degree 5.

Without loss of generality, we can assume $\ell = V(x_0, x_1, x_2) \subset \mathbb{P}^4_{x_0,\dots,x_4}$, and then we can write X as

$$\ell_1 x_3^2 + \ell_2 x_3 x_4 + \ell_3 x_4^2 + q_1 x_3 + q_2 x_4 + c = 0,$$

where the ℓ_i, q_j and c are homogenenous polynomials in $k[x_0, x_1, x_2]$ of degree 1, 2, and 3, respectively. Let

$$M := \begin{bmatrix} 2\ell_1 & \ell_2 & q_1 \\ \ell_2 & 2\ell_3 & q_2 \\ q_1 & q_2 & 2c \end{bmatrix}.$$

(1) Show that X' is isomorphic to the subscheme of $\mathrm{Proj}\, k[y_0, y_1, y_2] \times \mathrm{Proj}\, k[x, y, z]$ defined by the vanishing of

$$\ell_1(x,y,z)y_0^2 + \ell_2(x,z,y)y_0y_1 + \ell_3(x,y,z)y_1^2 + q_1(x,y,z)y_0y_2 + q_2(x,y,z)y_1y_2 + c(x,y,z)y_2^3.$$

(2) Show that the bundle map $\pi\colon X' \to \mathbb{P}^2$ is the restriction of projection $\mathrm{Proj}\, k[y_0, y_1, y_2] \times \mathrm{Proj}\, k[x, y, z] \to \mathrm{Proj}\, k[x, y, z]$, and that the fiber over a point $(a : b : c) \in \mathbb{P}^2$ is a conic with Gram matrix given by $M(a, b, c)$.
(3) Recall that a conic is singular if and only if its Gram matrix does not have full rank (think about this if it is not something you can recall!). Show that the discriminant divisor for $X' \to \mathbb{P}^2$ is a plane quintic curve.
(4) Show that the matrix M can be interpreted as a map of vector bundles $\mathcal{E} \to \mathcal{E}^\vee \otimes L$ for $\mathcal{E} = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-1)$ and $L = \mathcal{O}_{\mathbb{P}^2}(1)$.

From this perspective, $X' \subset \mathbb{P}(\mathcal{E})$ and moreover $\mathcal{E} = \pi_*\omega_\pi^{-1}$ for ω_π the relative canonical bundle of $\pi\colon X' \to \mathbb{P}^2$. This interprets the conic bundle as a line-bundle-valued quadratic form $q\colon \mathcal{E} \to L$ (see [ABB14, Lemma 1.1.1] for more).

References

[ABB14] Asher Auel, Marcello Bernardara, and Michele Bolognesi, *Fibrations in complete intersections of quadrics, Clifford algebras, derived categories, and rationality problems*, J. Math. Pures Appl. (9) **102** (2014), no. 1, 249–291, DOI 10.1016/j.matpur.2013.11.009 (English, with English and French summaries). MR3212256 ↑4

[GH94] Phillip Griffiths and Joseph Harris, *Principles of algebraic geometry*, Wiley Classics Library, John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original. MR1288523 ↑

[Har92] Joe Harris, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 133, Springer-Verlag, New York, 1992. A first course. MR1182558 ↑

[Har77] Robin Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York-Heidelberg, 1977. MR0463157 ↑1.6

[Huy23] Daniel Huybrechts, *The geometry of cubic hypersurfaces*, Cambridge Studies in Advanced Mathematics, vol. 206, Cambridge University Press, Cambridge, 2023. MR4589520 ↑2.4

[Poo17] Bjorn Poonen, *Rational points on varieties*, Graduate Studies in Mathematics, vol. 186, American Mathematical Society, Providence, RI, 2017. MR3729254 ↑

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