Problems on jet bundles

eriedl

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1 Day 2

1. Let $\mathbb{F}_e = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$.
   (a) Show that the Picard group of $\mathbb{F}_e$ is rank 2, generated by F, a fiber of the map to $\mathbb{P}^1$ and E, and ξ, the tautological class.
   (b) Show that ξ is effective but that $\mathcal{O}(m\xi)$ has a unique section for every $m > 0$.
   (c) Show that $\mathcal{O}(\xi + eF)$ is nef, and that a general element is disjoint from the divisor equivalent to ξ.
   (d) Show that $F \cdot F = 0$, $F \cdot \xi = 1$, $\xi^2 = -e$.
   (e) Compute the nef and effective cones of $\mathbb{F}_e$.
   (f) Let $S = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$. Then we see that S is isomorphic to $\mathbb{F}_3$. Find ξ_S in terms of the basic of the Picard group of $\mathbb{F}_3$.

2. Let X be a degree d surface in $\mathbb{P}^3$. The Chow ring of $\mathbb{P}^n$ is $A_k(\mathbb{P}^n) = H^k$, where H is the hyperplane class.
   (a) Compute $c(T_X)$ using

$$0 \to T_X \to T_{\mathbb{P}^3} \to \mathcal{O}(d) \to 0.$$

   (b) Find the Chow ring of $\mathbb{P}T_X$. Compute ξ^3.
   (c) Observe that for large d this means that $H^1(\mathcal{O}(m\xi))$ is much larger than $H^0(\mathcal{O}(m\xi))$. Show furthermore that ξ cannot be ample.
   (d) Compute $H\xi_1^2$ as a polynomial in d.

3. Consider the tangent bundle on $\mathbb{P}^2$.
   (a) Let $f = [f_0, f_1, f_2] : \mathbb{P}^1 \to \mathbb{P}^2$ be a degree e rational curve on $\mathbb{P}^2$, where the f_i are degree e polynomials with no common roots. (optional) Relate $f^*T_{\mathbb{P}^2}$ to the syzygies of the f_i.
   (b) Show that ξ in $\mathbb{P}T_{\mathbb{P}^2}$ is not effective by considering the restriction to a general line in $\mathbb{P}^2$.

(c) Show that $\xi + H$ is not nef by considering the restriction to a general line in $\mathbb{P}^2$.

(d) Show that $m\xi + (2m-1)H$ is not nef for any $m > 0$ by restricting to a general line.

(e) You will show in the day 3 exercises that $\xi + 2H$ is nef. Conclude that $\xi + 2H$ is on the boundary of the nef cone and find the other extremal ray.

Algebraic hyperbolicity of hypersurfaces (continued) By studying projective bundles over curves, we can prove algebraic hyperbolicity for $d = 2n - 1$ as well.

1. Consider the bundle M_1 on $\mathbb{P}^n$. Show that $M_1 = \Omega_{\mathbb{P}^n}(1)$. (Hint: compare the sequence defining M_1 to the Euler sequence.)

2. Let $f : C \to \mathbb{P}^n$ be a birational map from a smooth curve to a curve in $\mathbb{P}^n$ of degree e. Suppose f^*M_1 has a line bundle quotient L of degree $-k$. Show that this gives us a quotient Q' of $\mathcal{O}_C^{n+1}$ of rank 2 and degree $e - k$, compatible with the universal quotient $\mathcal{O}_C^{n+1} \to \mathcal{O}_C(1)$.

3. Taking duals, show that this gives us a rank 2 subsheaf Q'^* of $\mathcal{O}_C^{n+1}$ with degree $-e+k$, commuting with the universal subbundle $\mathcal{O}_C(-1) \to \mathcal{O}_C^{n+1}$. Thus, this gives us an inclusion $S = \mathbb{P}(Q'^*) \to C \times \mathbb{P}^n$.

4. Projecting S onto $\mathbb{P}^n$, this gives us a surface scroll in $\mathbb{P}^n$ containing C.

5. Show that the pullback of the $\mathcal{O}(1)$ on $\mathbb{P}^n$ to S has degree $e - k$.

6. Now suppose $f(C)$ lies on a degree d hypersurface X that doesn't contain any lines. Show by considering $S \cap X$ that $e \leq d(e - k)$.

7. Conclude that if $f(C)$ lies on a degree d hypersurface containing no lines, then any rank 1 quotiet of f^*M_1 has degree at least $-e(1 - \frac{1}{d})$.

8. Now recall the setting from Day 1. We have a surjection from a sum of s copies of f^*M_1 onto N_{h_b/X_b}. Show that if $s \leq n-3$, that $2g - 2 \geq \deg C$.

9. If $s = n-2$, then show that $N_{h_b/X_b}/(\mathrm{Im} f^*M_1^{s-1})$ is rank 1, and admits a generically surjective map from f^*M_1.

10. Using the discussion of scrolls, show that the degree of $N_{h_b/X_b}/(\mathrm{Im} f^*M_1^{s-1})$ is at least $-e(1 - \frac{1}{d})$.

11. Show that the degree of N_{h_b/X_b} is at least $-(n-2)e + \frac{e}{d}$.

12. Conclude that a general X of degree $d = 2n - 1$ in $\mathbb{P}^n$ is algebraically hyperbolic.

In fact, as Yeong shows in her thesis, using techniques of Voisin and Pacienza combined with this scroll technique, one can show algebraic hyperbolicity for $d = 2n - 2$ and $n \geq 5$. Since quartic surfaces are not algebraically hyperbolic, this leaves $n = 4$ as the only open case. Ask Wern Yeong for more details!