Problems on jet bundles

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June 2023

1 Day 3

1. (This is a repeat, but I'm guessing most people didn't get to it yesterday and it's a very good problem for setting up tomorrow's lecture.) Let X be a degree d surface in $\mathbb{P}^3$. The Chow ring of $\mathbb{P}^n$ is $A_k(\mathbb{P}^n) = H^k$, where H is the hyperplane class.
   (a) Compute $c(T_X)$ using
   $$0 \to T_X \to T_{\mathbb{P}^3} \to \mathcal{O}(d) \to 0.$$
   (b) Find the Chow ring of $\mathbb{P}T_X$. Compute ξ^3.
   (c) Observe that for large d this means that $H^1(\mathcal{O}(m\xi))$ is much larger than $H^0(\mathcal{O}(m\xi))$. Show furthermore that ξ cannot be ample.
   (d) Compute $H\xi_1^2$ as a polynomial in d.

2. Let $0 \to A \to B \to C \to 0$ be a short exact sequence of nef vector bundles. Suppose that A and C are nef. Show that B is also nef. (Hint: consider the $\mathcal{O}(1)$ on the projectivization of each of the spaces in turn.)

3. Show that if L is a line bundle and E is a vector bundle, then $\wedge^k(E \otimes L) = L^k \otimes \wedge^k E$.

4. Suppose $0 \to A \to B \to L \to 0$ is a short exact sequence of vector bundles with L a line bundle. Show that for $k \leq \mathrm{rk} A$ there is a short exact sequence
   $$0 \to \wedge^k A \to \wedge^k B \to \wedge^{k-1} A \otimes L \to 0.$$

5. Show that $\Omega_{\mathbb{P}^n}(2)$ is globally generated. (Hint: use the Euler sequence and take the second exterior power.)

Jet bundles for surfaces in $\mathbb{P}^3$ The following is the first of a series of exercises leading you through a study of the jet spaces of (general) surfaces in $\mathbb{P}^3$. This material is contained in papers of Demailly, Demailly-El Goul and Paun. These papers do some more complicated things, but the basic intersection theory calculation should be doable (and illustrative!) to those who attended the lectures.

1. This problem builds off of problem 2 from Day 2. Let X be a general degree d surface in $\mathbb{P}^3$. Consider the directed variety (X, T_X).

2. Compute the Chow ring of $X_1 = \mathbb{P}(T_X)$. (This should be the same calculation from yesterday, and it's worth doing if you haven't done it yet). To help yourself out for future parts of this problem, clearly write down what ξ_1^2 is and ξ_1^3 is.

3. Recall from lecture that we now wish to study the directed variety (X_1, V_1). Compute $c(V_1)$.

4. Write down the Chow ring of $X_2 = \mathbb{P}(V_1)$.

5. Compute ξ_2^2 (and write it clearly for future use).

6. Compute ξ_2^4. (Hint: it might make things a bit easier to first show that $\xi_1^3 H = 0$.)

7. Conclude using asymptotic Riemann-Roch that $\chi(\mathcal{O}(m\xi_2))$ grows very large for large m.

8. Let $E_{k,m}T_X^* = \pi_{k*}\mathcal{O}(m\xi_k)$, where π_k is the projection down to X. For $k = 2$, there is a filtration on $E_{2,m}T_X^*$ with quotients given by $\mathrm{Sym}^{m-3j}T_X^* \otimes K_X^j$, for $0 \leq j \leq \frac{m}{3}$. Using Bogomolov vanishing, show that this implies that $H^2(E_{2,m}T_X^*) = 0$. Using the fact that $\chi(\mathcal{O}(m\xi_2)) = \chi(\mathcal{O}(E_{2,m}T_X^*))$ (feel free to skip the justification of this fact), conclude that ξ_2 is big.

9. (optional challenge question) Suppose $a_2\xi_2 + a_1\xi_1 - \ell H$ is effective, with representative Z. Compute $(\xi_2+2\xi_1)^3|_Z$ to see how fast $\chi(\mathcal{O}(m(\xi_2+2\xi_1)))$ grows as m gets large. If χ grows like a cubic in m, it can be shown with Bogomolov vanishing (see Demailly-El Goul, Proposition 3.4) that $H^2(\mathcal{O}(m(\xi_2 + 2\xi_1)))$ vanishes.