Problems on jet bundles

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1 Day 4

1. Find the dimension of the Grassmannian $\mathbb{G}(k,n)$ of k-planes in $\mathbb{P}^n$. (Hint: Consider the space of tuples $(p_1,\dots,p_{k+1},\Lambda)$ and count its dimension.)
2. Let X be general type. Show that there exists an $\epsilon > 0$ such that any sweeping family of curves satisfies $2g-2 \geq C \cdot K_X$. (Hint: for a sweeping family of curves, the normal bundle will be globally generated.)
3. Consider the spaces X_k over a variety X.
    (a) Show that $\xi_k - \xi_{k-1}$ is effective for each k. (Hint: Consider the defining sequence of V_{k-1}, dualize, and take an appropriate twist.)
    (b) Show that the curves corresponding to this section are contracted by the map to X.
    (c) Given $p \in X_{k-2}$, describe the preimage of p in X_{k-1} as a projective space.
    (d) Now describe the preimage F of p in X_k as the projectivization of a vector bundle over your answer from the previous part.
    (e) Show that the section you of $\xi_k - \xi_{k-1}$ is the unique by considering the restriction to F for a general point p of X_{k-2}.
    (f) Conclude that there is a unique divisor D_k of class $\xi_k - \xi_{k-1}$ that is contracted by the map to X. We call this a divisor of stationary jets.
4. In this question, let X be a smooth hypersurface in $\mathbb{P}^n$. We will show that for a sequence of integers $a_1,\dots,a_k$ with $a_{k-1} \geq 2a_k$ and $a_i \geq 3a_{i+1}$ for $i < k-1$, and an integer $\ell \geq 2\sum_i a_i$, that $\ell H + \sum_{i=1}^k a_i\xi_i$ is nef.
    (a) First show $\xi_1 + 2H$ is nef. (Hint: consider the pushforward to X and use the result from yesterday's problem 5.)
    (b) Now let $a_k = 1$, $a_{k=1} = 2$ and $a_i = 2 \cdot 3^{k-i-1}$ for $i \leq k-1$. Consider the divisors $L_k = \sum_{i=1}^k a_i\xi_i + 2 \cdot 3^{k-1}H$ and $A_k = L_k - \xi_k$. Show for $k = 1$ that L_1 and A_1 are nef.
    (c) Show that $A_{k+1} = 3A_k + 2\xi_k = A_k + 2L_k$.

(d) We now use induction. Suppose that L_k and A_k are nef. Show that this implies A_{k+1} is nef.

(e) It remains to show that L_{k+1} is nef. By considering $\pi_* L_{k+1}$, show that it is enough to show $V_k^*(A_{k+1})$ is nef.

(f) Using the defining sequence of V_k, show that it suffices to show that $\mathcal{O}(3A_k + 3\xi_k))$ and $T^*_{\pi_k}(3A_k + 2\xi_k)$ are nef.

(g) Show that $\mathcal{O}(3A_k + 3\xi_k)$ is nef by relating it to L_k.

(h) Show that $T^*_{\pi_k}(3A_k + 2\xi_k)$ is nef by taking the dual of the second exterior power of the Euler sequence of V_{k-1}. (Hint: you will need to twist the result you get and use induction, remembering that $V^*_{k-1}(A_{k+1}) = \pi_* L_k$.)

(i) Conclude that A_k and L_k are always nef.

(j) Conclude the result with inequlities in the setup using what you have shown already.

5. Suppose that X is rationally connected. Show that ξ_k is never big for any k.