Problems on jet bundles

eriedl

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1 Day 1

Basic practice

1. Let D be a big divisor and E be an effective divisor. Show that $D+E$ is big.

2. Recal that a divisor D is nef if its restriction to every curve C in X has non-negative degree. Find an example of a divisor that is nef but not ample. (Hint: there's one on $\mathbb{P}^1 \times \mathbb{P}^1$.

3. Find an example of a divisor that is big but not nef. (Hint: there's one on the blowup of $\mathbb{P}^2$.

4. Suppose X is a complete intersection of hypersurfaces in $\mathbb{P}^n$ of type $(d_1, d_2, \ldots, d_k)$. Find the canonical sheaf K_X. (Hint: use the adjunction formula.)

5. Let L and L' be line bundles on a smooth curve C, and suppose there exists a nonzero map $L \to L'$. Show that $\deg L \geq \deg L'$, with equality if and only if L is isomorphic to L'.

6. Let $C \subset X$ be a smooth curve in X. Show that the degree of the normal bundle $N_{C/X}$ is $2g-2-K_X \cdot C$.

7. Let $h : C \to X$ be a birational map from a smooth curve to X. Define $N_{h/X}$ as the quotient of h^*T_X by T_C. Show that $\deg N_{h/X}$ is $2g-2-K_X \cdot C$.

8. For all positive integers d, define the syzygy bundle M_d on $\mathbb{P}^n$ via the following sequence:

$$0 \to M_d \to H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \otimes \mathcal{O} \to \mathcal{O}(d) \to 0.$$

   Show that $H^0(M_d) = 0 = H^1(M_d)$.

9. Let $f : C \to \mathbb{P}^n$ be a smooth, degree e curve mapping birationally. Show that every quotient of f^*M_1 has degree at least $-e$. Show further that every quotient of a direct sum of s copies of f^*M_1 has degree at least $-se$.

Algebraic hyperbolicity of hypersurfaces The following series of problems takes you a proof of the algebraic hyperbolicity of general hypersurfaces in $\mathbb{P}^n$ of degree $d \geq 2n-1$. This problem has a long history, with the original idea dates back to work from Clemens and Ein, which was elaborated on by Voisin, Xu, Pacienza, Clemens-Ran, and Riedl-Coskun. This version most closely resembles the more streamlined presentation from Wern Yeong's thesis.

1. (Setup) Suppose that a general hypersurface X in $\mathbb{P}^n$ contains a curve of degree e and geometric genus g. Let $\mathcal{X} \subset \mathbb{P}^n \times H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ be the space of pairs (f, p) such that p in a point in the hypersurface $V(f)$. Consider the space M' of maps from smooth genus g curves to $\mathcal{X}$ such that the image is a degree e curve in the fiber of $\mathcal{X} \to H^0(\mathcal{O}_{\mathbb{P}^n}(d))$. By hypothesis, M' dominates $H^0(\mathcal{O}_{\mathbb{P}^n}(d))$.
   (a) Show that there exists a subvariety B of M' such that M parameterizes only finitely curves for a general hypersurface.
   (b) Show that we can select B to be a PGL-invariant family.
   (c) Show that by possibly restricting to an open set, we can assume the map $B \to H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ is etale.
   (d) Base-change $\mathcal{X}$ to B, so that we have a family of curves $\mathcal{Y} \to B$ and a map $h : \mathcal{Y} \to \mathcal{X}_B$. From now one, we write $\mathcal{X}$ for $\mathcal{X}_B$.
2. Let Y_b be a general fiber of $\mathcal{Y} \to B$ and X_b the fiber of $\mathcal{X}$ over b, with map $h_b : Y_b \to X_b$. Show that $N_{h/\mathcal{X}}|_{Y_b} = N_{h_b/X_b}$. (Hint: write down a commutative diagram with the relevant pieces and use the eight lemma.)
3. Recall we have the relative tangent sheaves $T_{\mathcal{X}/\mathbb{P}^n}$ and $T_{\mathcal{Y}/\mathbb{P}^n}$, given by the kernels of $T_{\mathcal{X}\to\mathbb{P}^n}$ and $T_{\mathcal{Y}} \to \mathbb{P}^n$. Show that $N_{h/\mathcal{X}}$ is the quotient of $h^*T_{\mathcal{X}/\mathbb{P}^n}$ by $T_{\mathcal{Y}/\mathbb{P}^n}$. (Hint: this is another diagram chase.)
4. Show that $T_{\mathcal{X}/\mathbb{P}^n}$ is isomorphic to the pullback of M_d from $\mathbb{P}^n$. (Hint: this is another diagram chase, using the fact that $T_B|_{X_b} = H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \otimes \mathcal{O}$.)
5. Show that given any degree $d-1$ polynomial P, we have a natural map $M_1 \to M_d$ given by multiplication by P.
6. Show that there is a surjection from a direct sum of many copies of M_1 to M_d.
7. Show that this implies that there is a surjective map from a direct sum of copies of $h_b^*M_1$ to N_{h_b/X_b}.
8. Show that in fact there is a generically surjective map from a direct sum of at most $n-2$ copies of $h_b^*M_1$ onto N_{h_b/X_b}, and that the degree of N_{h_b/X_b} is at least $-(n-2)e$.
9. Conclude that X must be algebraically hyperbolic for $d \geq 2n$.
10. Show that for $d \leq 2n-3$, X must contain a line, leaving open only the cases $d = 2n-1, 2n-2$.