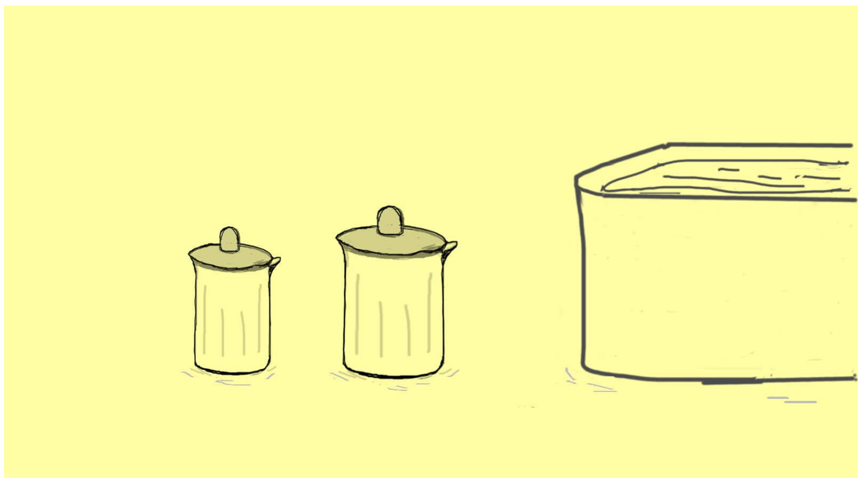
Lecture 1: Jug problem

We start with a classical problem.

Problem 0.1. Suppose you are standing next to a large pool of water. You have two jugs – a 5 liter jug and a 7 liter jug. You would like to measure out exactly 4 liters of water. Can you do it?



Note that you can use the jugs in combination – for example, if you fill the 7 liter jug and then use it to fill the 5 liter jug, you will have 2 liters left over in the 7 liter jug. However, we are also supposed to imagine the jugs are "irregularly shaped" so it is not possible to measure out e.g. half of a jug.

1) Suppose we have jugs of size 5 liters and 7 liters. Can you measure out 4 liters? 1 liter? What amounts can we measure using these two jugs? (Try keeping a list!)

2) What if we instead use jugs of size 4 liters and 10 liters; what are the possibilities now?

3) Try some other combinations. Given the sizes of the two jugs, can you predict which amounts are measurable?

4) Can you find the "shortest solution" requiring the least amount of fill-ups and pour-outs if you have jugs of size 5 and 7? In general?

Let's discuss how to model this problem mathematically. Let x represent the number of times we fill up the 5 liter jug and y denote the number of times we fill up the 7 liter jug.

(We allow these numbers to be negative; this represents emptying a jug back into the pool.) Then the total amount of water we have measured out is

$$5x + 7y$$

where x and y are integers. Our problem is roughly equivalent to asking: what are the possible values of $5x + 7y$ as we let x, y vary over all integers.

Exploratory Question 0.2. There are some serious flaws with our mathematical model. For example, we would need to think carefully about whether we can actually reach a certain sum $5x + 7y$ by successively filling up and pouring out our two jugs. You might enjoy thinking about the limitations of our model and whether we can use it to answer our original problem.

More generally, suppose we have two jugs with sizes A, B. This set of exercises will help you work through the following claim:

Theorem 0.3. *The set of integers of the form $Ax + By$ as we vary $x, y \in \mathbb{Z}$ is exactly the set of all multiples of* $\gcd(A, B)$.

For example, since $\gcd(4, 10) = 2$ the set of integers of the form $4x + 10y$ is exactly the even integers: $\ldots, -4, -2, 0, 2, 4, 6, \ldots$. As we'll see, the hard part of the theorem above is to show that $\gcd(A, B)$ can be written as a combination of A and B.

5) Show that $\gcd(A, B)$ divides every combination $Ax + By$.

6) Show that every number d that divides both A and B will also divide $A - B$. In particular, prove that $\gcd(A, B) = \gcd(A - B, B)$.

7) Suppose we divide A by B with remainder to write $A = qB + r$ where $0 \leq r < A$. Prove the following claims:

   - If $r \neq 0$ we have $\gcd(A, B) = \gcd(B, A - qB) = \gcd(B, r)$.
   - If $r = 0$ then $\gcd(A, B) = B$.

8) (Euclidean Algorithm) Suppose we do the following inductive procedure. We set $a_0 = A$ and $b_0 = b$. We then divide with remainder to write $a_0 = q_0 b_0 + r_0$. We then define $a_1 = b_0$, $b_1 = r$. The previous problem shows that either $r_0 = 0$ (in which case b_0 is the gcd) or $r \neq 0$ and we have $\gcd(a_0, b_0) = \gcd(a_1, b_1)$. Furthermore we have the key property: $b_1 < b_0$.

   We now repeat the procedure: we divide with remainder to get $a_1 = q_1 b_1 + r_1$ and set $a_2 = b_1$, $b_2 = r_1$. Then either $r_1 = 0$, in which case $b_1 = \gcd(a_1, b_1) = \gcd(a_0, b_0)$, or $b_2 < b_1$ and we repeat the process with our new a_2, b_2.

   We continue this repeated division under we get remainder 0. Show that this procedure will terminate and that the last (non-zero) value b_k will be the gcd of our two original numbers (A, B).

9) Using the previous problem, explain why $\gcd(A, B)$ can always be written as a combination of A, B.

10) Finally, put everything together to prove Theorem 0.3.

Exploratory Question 0.4. Imagine we have three jugs of sizes A, B, C. What amounts can we measure now? How does the situation change? Can you develop a theory of multiple jugs that is analogous to the theory outlined above?

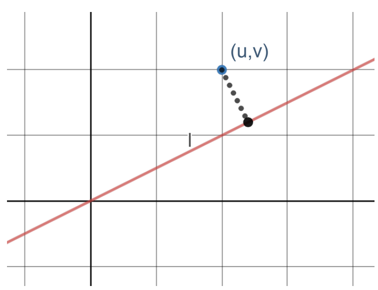
Exploratory Question 0.5. Once we know that $Ax + By = c$ has a solution, it is natural to ask: what are all solutions to this equation? Is there a "smallest" solution? How can you find it?

When the equation $Ax + By = c$ has no solution, we can still ask how "close" we can get. To help us understand what "close" means, we will start modeling this problem using lines. If we draw the line $Ax + By = c$ in the plane, the line may or may not go through any integer points $\mathbb{Z}^2 \subset \mathbb{R}^2$. If it does, our equation has a solution! If not, we can look for an "approximate" solution:

11) The line $2x + 4y = 3$ does not contain any points in $\mathbb{Z}^2$. Which lattice point(s) are closest to the line? How do you know?

12) The line $12x + 15y = 7$ does not contain any points in $\mathbb{Z}^2$. Which lattice point(s) are closest to the line? How do you know?

13) Suppose we have a line ℓ with equation $ax + by = c$ where a, b, c are integers. Suppose that $(u, v) \in \mathbb{Z}^2$ is a point with integer coordinates. What is the formula for the (perpendicular) distance from (u, v) to the line ℓ?



Distance from a point to a line

14) Which point $(u, v) \in \mathbb{Z}^2$ is closest to ℓ? Use the formula to explain your answer.