# Lecture 1: Rational points on conics

Let's first remember our strategy for finding the integer solutions to $a^2 + b^2 = c^2$, or (more-or-less) equivalently, the rational solutions to $x^2 + y^2 = 1$:

- We first find a single rational point $p$ on the circle $x^2 + y^2 = 1$.
- We take the line through $p$ with slope $t$ and, using a factoring technique, we find the other point of intersection $q_t$ of this line with the circle.
- As we vary over rational slopes $t$, the points $q_t$ fill out (almost) all the rational points on our circle.

Let's try this strategy in some new situations. Suppose we are looking for all triples of integers $(a, b, c)$ such that

$$a^2 + 2b^2 = c^2.$$

Note that the order matters, since the equation is not symmetric in $a$ and $b$. Some primitive examples are $(1, 2, 3)$ and $(7, 4, 9)$. It is not so easy to find more just by looking at the equation!

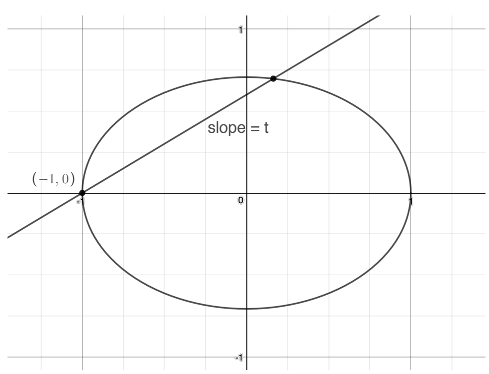
By dividing both sides of the equation by $c^2$, we see that finding primitive triples as above is the same as finding rational points on the ellipse

$$x^2 + 2y^2 = 1.$$

We will fix the point $p = (-1, 0)$ on this ellipse.

1) Prove that the line through $(-1, 0)$ of slope $t$ will also meet the ellipse at the point

$$\left(\frac{1-2t^2}{1+2t^2}, \frac{2t}{1+2t^2}\right)$$



2) Conclude that we can generate all primitive triples solving our original equation by plugging in rational numbers $t$ to the parametric formula above and clearing denominators.

3) Find the rational points corresponding to the $t$-values $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, and $\frac{1}{4}$.

4) Find the $t$-values corresponding to the triples $(7, 30, 43)$ and $(41, 28, 57)$.

We can run the same argument for *any* conic. Once a (smooth) conic has a single rational point, it is guaranteed to have infinitely many of them and we can find them all by constructing a rational parametrization. Let's try a few more.

5) Fix a positive integer $n$ and consider the triples of integers $(a, b, c)$ satisfying $a^2 + nb^2 = c^2$. Show that the rational parametrization of the ellipse $x^2 + ny^2 = 1$ which assigns to any point the slope of the line connecting it to $(-1, 0)$ is given by

$$\left(\frac{1 - nt^2}{1 + nt^2}, \frac{2t}{1 + nt^2}\right).$$

   Use this to identify some integer solutions to the equation.

6) Find all the rational points on $x^2 + y^2 = 2$.

7) Show that there are no rational points on the circle $x^2 + y^2 = 3$. (Hint: first rescale to get an equation in integers $a^2 + b^2 = 3c^2$. Try looking at remainders mod 4 and doing a descent argument.)

8) Show that there are no rational points on the ellipse $2x^2 + 3y^2 = 1$.

9) The equation $2x^2 + 7y^2 = 1$ does have rational points; for example, the point $(\frac{-1}{3}, \frac{-1}{3})$. Project away from the point $(\frac{-1}{3}, \frac{-1}{3})$ to find all the rational points.

10) Pell's equation has the form $x^2 - Dy^2 = 1$ where $D$ is an integer $> 1$. This equation always has the solution $(-1, 0)$, so we can find all the rational solutions using the techniques described earlier.

    a) Find all the rational solutions to Pell's equation.
    b) Show that if $D$ is a perfect square, then the equation has no *integer* solutions besides the trivial ones $(\pm 1, 0)$.

    It turns out that when $D$ is not a perfect square Pell's equation has infinitely many *integer* solutions. (Can you see this from the rational parametrization?) Pell's equation plays an important role in number theory due to its connections with $\sqrt{D}$. Note that for any integer solution $(x, y)$ the quotient $\frac{x}{y}$ is "very close" to $\sqrt{D}$. In

fact, using the theory of continued fractions one can show that such rational numbers $\frac{x}{y}$ are the "best approximations" to $\sqrt{D}$. One can also study Pell's equation using techniques from algebraic number theory by factoring
$x^2 - Dy^2 = (x + \sqrt{D}y)(x - \sqrt{D}y)$ inside of $\mathbb{Z}[\sqrt{D}]$.