Lecture 2: Area and lattice points

For us a lattice point in $\mathbb{R}^2$ will mean a point (x, y) with integer coordinates. In lecture we proved the following result:

Theorem 0.1 (Gauss)**.** *Let $N(r)$ denote the number of lattice points contained in the circle of radius r centered at the origin. Then*

$$\pi r^2 - \sqrt{2}\pi r + \frac{1}{2} \leq N(r) \leq \pi r^2 + \sqrt{2}\pi r + \frac{1}{2}$$

This is just one example of a large collection of theorems relating the area of a shape $S \subset \mathbb{R}^2$ to the number of lattice points it contains. In this problem set we will explore some of these results.

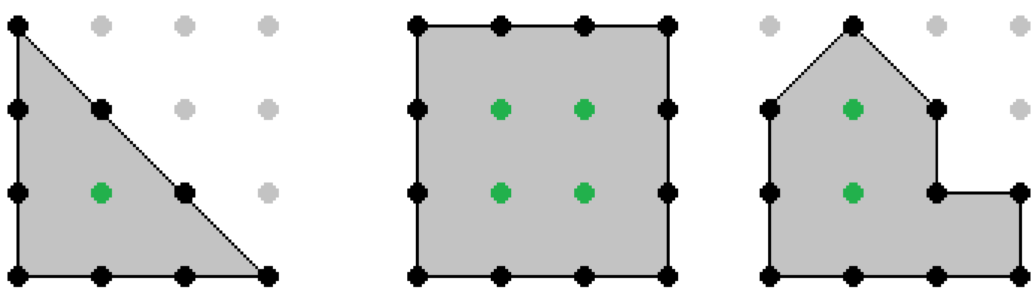
1 Pick's Theorem

A (non-self intersecting) polygon $P \subset \mathbb{R}^2$ is a shape whose sides are straight line segments which only meet along their endpoints. Pick's Theorem relates the area of a polygon P with the number of points it contains:

Theorem 1.1. *Suppose that $P \subset \mathbb{R}^2$ is a polygon whose vertices lie on lattice points. Let I denote the number of lattice points contained in the interior of P and let B denote the number of lattice points contained on the boundary of P. Then*

$$\textit{area}(P) = I + \frac{B}{2} - 1.$$

1) Verify Pick's Theorem for the following shapes:



2) Prove Pick's Theorem for a rectangle whose sides are parallel to the x and y axes.

3) Prove Pick's Theorem for a right triangle whose short sides are parallel to the x and y axes.

4) Prove Pick's Theorem for every triangle. (Hint: enclose the triangle in a rectangle whose sides are parallel to the x and y axes. Use the previous problem to compute the difference in areas between the rectangle and the original triangle.)

5) Prove Pick's Theorem. (Hint: use induction to show that one can divide any polygon whose vertices are lattice points into triangles and apply the previous result.)

Exploratory Question 1.2. There is no exact analogue of Pick's Theorem in three dimensions. For example, consider the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, k)$. This tetrahedron always contains four boundary points and no interior points. However, its volume can change depending on the value of k!

For polyhedra in dimensions ≥ 3, there is a more complicated relationship between volume and lattice points described by the theory of Ehrhart polynomials. Look it up if you are curious!

2 Minkowski's Theorem

While Pick's Theorem is very satisfying, it only addresses a very special type of shape. It is interesting to ask if there is any similar result for more general shapes. Of course, we need to have some assumption – one can always draw a random shape with large area that avoids every lattice point!
Minkowski's Theorem addresses shapes $S \subset \mathbb{R}^2$ that have the following properties:

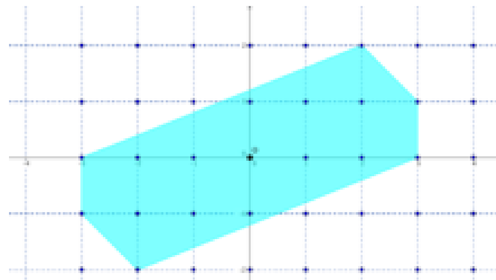
1) S is bounded, i.e. it is contained in a rectangle whose sides have finite length.

2) S is convex: if $p, q \in S$ then every point on the line segment between p and q is in S.

3) S is centrally symmetric: S contains the origin and if $(x, y) \in S$ then $(-x, -y) \in S$ as well.

Remark 2.1. Every convex set has several other desirable properties – for example they are measurable and their boundary is differentiable away from a set of measure zero – guaranteeing that the various quantities we use later on are well-defined.

Theorem 2.2. *Let $S \subset \mathbb{R}^2$ be a bounded, convex, centrally symmetric set. If area$(S) > 4$ then S contains at least one lattice point different from the origin.*



A bounded, convex, centrally symmetric shape

Note that the constant 4 is the best we can do: if we take a square of side length $2 - \epsilon$ centered at the origin, it will have area a little less than 4 but won't contain any lattice points besides the origin.

Let's set up the proof. We tile the plane by all the squares of side length 2 whose vertices are pairs of odd integers. A particularly important role will be played by the square of side length 2 centered at the origin – we call this square Q.
Since S is bounded, it will only intersect a finite number of these squares $Q_1, \ldots, Q_k$. We imagine taking the sets $S \cap Q_k$ and translating the squares Q_k back to Q to get sets $T_1, T_2, \ldots, T_k$ contained in Q.

6) Explain why there are two different indices $1 \leq i, j \leq k$ such that the corresponding sets T_i, T_j intersect. Conclude that for some point $t \in Q$ there are different vectors v, w whose coordinates are even integers such that $t + v$ and $t + w$ are both in S.

7) Using the fact that S is symmetric and convex, conclude that $\frac{v-w}{2}$ is a lattice point in S that is different from the origin.

8) Develop and prove a version of Minkowski's Theorem in $\mathbb{R}^n$.

3 Asymptotic area

While the previous results are interesting, they are a little different from Gauss' circle problem. To obtain a better analogy, we ask the following: suppose that $S \subset \mathbb{R}^2$ is a bounded convex set that contains the origin. Given a positive number r, we denote by rS the shape obtained by rescaling all the vectors in S by the constant factor r. Let $N(r)$ denote the number of lattice points in rS. We would like to understand how the function $N(r)$ behaves as r grows to infinity.

We first need an auxiliary definition:

Definition 3.1. Let $S \subset \mathbb{R}^2$ be a bounded convex set containing the origin. We define

$$V(S) = \inf_B \{\text{perim}(B) + 4\}$$

as we vary B over all rectangles containing S whose sides are parallel to the x and y axes.

The following theorem gives an analogue of Gauss' result for circles.

Theorem 3.2. *Given a region $S \subset \mathbb{R}^2$ as above, we have*

$$r^2 \text{area}(S) - rV(S) \leq N(r) \leq r^2 \text{area}(S) + rV(S).$$

9) Use Pick's Theorem to verify Theorem 3.2 when S is a polygon whose vertices are lattice points.

We prove Theorem 3.2 by mimicking our approach to Gauss' circle problem. Let T_r denote the union of the squares of side length 1 centered at the points $p \in rS$.

10) Explain why the area of T_r is the same as $N(r)$.

11) Let U_r^+ denote the union of all lattice squares which contain a point of rS and let U_r^- denote the union of all lattice squares which are entirely contained in rS. Explain why $U_r^- \subset T_r \subset U_r^+$.

12) By comparing rS to a rectangle containing it, explain why $\text{area}(rS) - V(rS) \leq \text{area}(U_r^-)$ and $\text{area}(rS) + V(rS) \geq \text{area}(U_r^+)$.

13) Prove Theorem 3.2.

Exploratory Question 3.3. Try generalizing Gauss' circle problem to higher dimensions. For example, approximately how many lattice points are contained in a sphere of radius r in $\mathbb{R}^3$? In a sphere of radius r in $\mathbb{R}^n$?