Lecture 2: Area and lattice points

For us a lattice point in $\mathbb{R}^2$ will mean a point $(x, y)$ with integer coordinates. In lecture we proved the following result:

**Theorem 0.1** (Gauss). Let $N(r)$ denote the number of lattice points contained in the circle of radius $r$ centered at the origin. Then

$$
\pi r^2 - \sqrt{2} \pi r + \frac{1}{2} \leq N(r) \leq \pi r^2 + \sqrt{2} \pi r + \frac{1}{2}
$$

This is just one example of a large collection of theorems relating the area of a shape $S \subset \mathbb{R}^2$ to the number of lattice points it contains. In this problem set we will explore some of these results.

1 Pick’s Theorem

A (non-self intersecting) polygon $P \subset \mathbb{R}^2$ is a shape whose sides are straight line segments which only meet along their endpoints. Pick’s Theorem relates the area of a polygon $P$ with the number of points it contains:

**Theorem 1.1.** Suppose that $P \subset \mathbb{R}^2$ is a polygon whose vertices lie on lattice points. Let $I$ denote the number of lattice points contained in the interior of $P$ and let $B$ denote the number of lattice points contained on the boundary of $P$. Then

$$
\text{area}(P) = I + \frac{B}{2} - 1.
$$

1) Verify Pick’s Theorem for the following shapes:

2) Prove Pick’s Theorem for a rectangle whose sides are parallel to the $x$ and $y$ axes.
3) Prove Pick’s Theorem for a right triangle whose short sides are parallel to the $x$ and $y$ axes.

4) Prove Pick’s Theorem for every triangle. (Hint: enclose the triangle in a rectangle whose sides are parallel to the $x$ and $y$ axes. Use the previous problem to compute the difference in areas between the rectangle and the original triangle.)

5) Prove Pick’s Theorem. (Hint: use induction to show that one can divide any polygon whose vertices are lattice points into triangles and apply the previous result.)

**Exploratory Question 1.2.** There is no exact analogue of Pick’s Theorem in three dimensions. For example, consider the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,k)$. This tetrahedron always contains four boundary points and no interior points. However, its volume can change depending on the value of $k$!

For polyhedra in dimensions $\geq 3$, there is a more complicated relationship between volume and lattice points described by the theory of Ehrhart polynomials. Look it up if you are curious!

## 2 Minkowski’s Theorem

While Pick’s Theorem is very satisfying, it only addresses a very special type of shape. It is interesting to ask if there is any similar result for more general shapes. Of course, we need to have some assumption – one can always draw a random shape with large area that avoids every lattice point!

Minkowski’s Theorem addresses shapes $S \subset \mathbb{R}^2$ that have the following properties:

1) $S$ is bounded, i.e. it is contained in a rectangle whose sides have finite length.

2) $S$ is convex: if $p, q \in S$ then every point on the line segment between $p$ and $q$ is in $S$.

3) $S$ is centrally symmetric: $S$ contains the origin and if $(x, y) \in S$ then $(-x, -y) \in S$ as well.

**Remark 2.1.** Every convex set has several other desirable properties – for example they are measurable and their boundary is differentiable away from a set of measure zero – guaranteeing that the various quantities we use later on are well-defined.

**Theorem 2.2.** Let $S \subset \mathbb{R}^2$ be a bounded, convex, centrally symmetric set. If $\text{area}(S) > 4$ then $S$ contains at least one lattice point different from the origin.
A bounded, convex, centrally symmetric shape

Note that the constant 4 is the best we can do: if we take a square of side length $2 - \epsilon$ centered at the origin, it will have area a little less than 4 but won’t contain any lattice points besides the origin.

Let’s set up the proof. We tile the plane by all the squares of side length 2 whose vertices are pairs of odd integers. A particularly important role will be played by the square of side length 2 centered at the origin – we call this square $Q$.

Since $S$ is bounded, it will only intersect a finite number of these squares $Q_1, \ldots, Q_k$. We imagine taking the sets $S \cap Q_k$ and translating the squares $Q_k$ back to $Q$ to get sets $T_1, T_2, \ldots, T_k$ contained in $Q$.

6) Explain why there are two different indices $1 \leq i, j \leq k$ such that the corresponding sets $T_i, T_j$ intersect. Conclude that for some point $t \in Q$ there are different vectors $v, w$ whose coordinates are even integers such that $t + v$ and $t + w$ are both in $S$.

7) Using the fact that $S$ is symmetric and convex, conclude that $\frac{v-w}{2}$ is a lattice point in $S$ that is different from the origin.

8) Develop and prove a version of Minkowski’s Theorem in $\mathbb{R}^n$.

3 Asymptotic area

While the previous results are interesting, they are a little different from Gauss’ circle problem. To obtain a better analogy, we ask the following: suppose that $S \subset \mathbb{R}^2$ is a bounded convex set that contains the origin. Given a positive number $r$, we denote by $rS$ the shape obtained by rescaling all the vectors in $S$ by the constant factor $r$. Let $N(r)$ denote the number of lattice points in $rS$. We would like to understand how the function $N(r)$ behaves as $r$ grows to infinity.

We first need an auxiliary definition:
Definition 3.1. Let $S \subset \mathbb{R}^2$ be a bounded convex set containing the origin. We define

$$V(S) = \inf_B \{ \text{perim}(B) + 4 \}$$

as we vary $B$ over all rectangles containing $S$ whose sides are parallel to the $x$ and $y$ axes.

The following theorem gives an analogue of Gauss’ result for circles.

Theorem 3.2. Given a region $S \subset \mathbb{R}^2$ as above, we have

$$r^2 \text{area}(S) - rV(S) \leq N(r) \leq r^2 \text{area}(S) + rV(S).$$

9) Use Pick’s Theorem to verify Theorem 3.2 when $S$ is a polygon whose vertices are lattice points.

We prove Theorem 3.2 by mimicking our approach to Gauss’ circle problem. Let $T_r$ denote the union of the squares of side length 1 centered at the points $p \in rS$.

10) Explain why the area of $T_r$ is the same as $N(r)$.

11) Let $U_{r}^+$ denote the union of all lattice squares which contain a point of $rS$ and let $U_{r}^-$ denote the union of all lattice squares which are entirely contained in $rS$. Explain why $U_{r}^- \subset T_r \subset U_{r}^+$.

12) By comparing $rS$ to a rectangle containing it, explain why $\text{area}(rS) - V(rS) \leq \text{area}(U_{r}^-)$ and $\text{area}(rS) + V(rS) \geq \text{area}(U_{r}^+)$.

13) Prove Theorem 3.2.

Exploratory Question 3.3. Try generalizing Gauss’ circle problem to higher dimensions. For example, approximately how many lattice points are contained in a sphere of radius $r$ in $\mathbb{R}^3$? In a sphere of radius $r$ in $\mathbb{R}^n$?