Lecture 4: Gauss-Bonnet

This set of exercises will give you more practice with the Gauss-Bonnet formula.

Let’s start by remembering how to define the curvature for a smooth oriented parametric surface \( S \subset \mathbb{R}^3 \). (Here oriented means we have a compatible choice of direction for the normal vectors at every point.) Suppose our parametrization is given by a function \( f: \mathbb{R}^2 \to \mathbb{R}^3 \):

\[
(\mathbf{f}(u,v)) = (f_1(u,v), f_2(u,v), f_3(u,v)).
\]

As in lecture, we denote by \( f_u, f_v, f_{uu}, f_{uv}, f_{vv} \) the vectors obtained by taking the single/double derivatives of the components of \( f \). Finally, we also consider the normal unit vector

\[
\hat{n} = \frac{f_u \times f_v}{\|f_u \times f_v\|}.
\]

(To be precise we should ensure that the direction of \( \hat{n} \) agrees with our orientation.) Then the Gaussian curvature of \( S \) at a point \( p \) is given by the formula

\[
K = \frac{\langle f_{uu}, \hat{n} \rangle \langle f_{vv}, \hat{n} \rangle - \langle f_{uv}, \hat{n} \rangle^2}{\langle f_u, f_u \rangle \langle f_v, f_v \rangle - \langle f_u, f_v \rangle^2}.
\]

Now suppose that \( S \subset \mathbb{R}^3 \) is a smooth compact oriented surface.

**Theorem 0.1 (Gauss-Bonnet).** Let \( S \) be a smooth compact orientable surface with no boundary. Then

\[
\int_S K \, dA = 2\pi \chi(S)
\]

where \( dA \) is the area element on \( S \).

Let’s try some examples!

Cross section: major radius \( c \), minor radius \( a \)

First consider the torus parametrized by

\[
f(u,v) = ((c + a \cos(v)) \cos(u), (c + a \cos(v)) \sin(u), a \sin(v))
\]

for the range \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 2\pi \). (Here \( u \) represents an angle in the \( x, y \)-plane and \( v \) represents the angle in the “small circle” formed by taking a cross-section of the torus.)
1) Verify carefully that the curvature satisfies

\[ K(u,v) = \frac{\cos(v)}{a(c + a\cos(v))} \]

2) Verify that the Gauss-Bonnet formula holds for the parametrized torus above.

Unfortunately it becomes much more difficult to parametrize surfaces of genus \( \geq 2 \). Instead we will go in a slightly different direction.

1 Gauss-Bonnet for surfaces with boundary

An important feature of the original Gauss-Bonnet formula is that it is stated for surfaces “without boundary”: every point has an open neighborhood that is homeomorphic to an open disk in \( \mathbb{R}^2 \). In a surface “with boundary”, every point has an open neighborhood that is homeomorphic to an open ball in the closed upper half-plane in \( \mathbb{R}^2 \).

Surfaces without boundary / with boundary

There is also a version of the Gauss-Bonnet formula that holds for surfaces with boundary. In this situation we will need to include a correction term to account for the removed region. This correction term comes from a quantity known as “geodesic curvature.” Let’s suppose the boundary curve is parametrized by a function \( g \):

\[ g(t) = (g_1, g_2, g_3). \]

We need to insist that the orientation of the parametrization is compatible with our surface orientation. This is determined by the right hand rule: suppose we choose a point
Given a parametrized curve $g$ as above, the geodesic curvature is defined as

$$k_g = \frac{1}{\|g'(t)\|^2} g''(t) \cdot (\hat{n} \times \hat{t})$$

where $\hat{n}$ is the unit normal vector to the surface and $\hat{t}$ is the unit tangent vector to the curve.

**Remark 1.1.** A curve in $S$ is said to be a geodesic if $k_g = 0$. Such curves are the “shortest paths” between two points in the surface $S$!

**Example 1.2.** Consider the circle $(\cos(t), \sin(t), 0)$ in the $x,y$-plane. We can think of this as the boundary of a flat region in the $x,y$-plane. We will orient this surface in the upward direction, so the unit normal is the constant vector $(0,0,1)$. (Note that our choice of orientation of the surface is compatible with our choice of orientation of the boundary via the right hand rule.) Therefore $\hat{n} \times \hat{t}$ is the vector $(0,0,1) \times (-\sin(t), \cos(t), 0) = (-\cos(t), -\sin(t), 0)$. Then the geodesic curvature is

$$k_g = \frac{1}{\sin(t)^2 + \cos(t)^2} (-\cos(t), -\sin(t), 0) \cdot (-\cos(t), -\sin(t), 0)$$

The various terms cancel out to give $k_g = 1$.

**Example 1.3.** Consider the circle $(\cos(t), \sin(t), 0)$ in the $x,y$-plane. This time we think of the circle as the boundary of the “spherical cap” lying over the circle. We will orient this surface in the upward direction; this means that the unit surface normal along the circle is given by $(\cos(t), \sin(t), 0)$. (Note that our choice of orientation of the surface is compatible with our choice of orientation of the boundary via the right hand rule.) Therefore $\hat{n} \times \hat{t}$ is the vector $(\cos(t), \sin(t), 0) \times (-\sin(t), \cos(t), 0) = (0,0,1)$. Then the geodesic curvature is

$$k_g = \frac{1}{\sin(t)^2 + \cos(t)^2} (-\sin(t), -\cos(t), 0) \cdot (0,0,1)$$

In this case we have $k_g = 0$.

We can now state a version of Gauss-Bonnet for surfaces with piecewise smooth boundary:
**Theorem 1.4 (Gauss-Bonnet).** Let $S$ be a smooth compact orientable surface with a piecewise smooth boundary which has corners at $\{v_i\}_{i=1}^r$. Then

$$
\int_S K \, dA + \int_{\partial S} k_g \, ds + \sum_{i=1}^r \theta_{v_i} = 2\pi \chi(S)
$$

where $dA$ is the area element on $S$, $ds$ is the line element along the boundary $\partial S$, and $\theta_{v_i}$ denotes the exterior angle at $v_i$.

Remember, to compute the interior angle $\alpha$ between two smooth curves meeting at a point $p$, one computes their tangent vectors $t_1, t_2$ at $p$ and computes the angle between them:

$$
\cos(\alpha) = \frac{t_1 \cdot t_2}{\|t_1\| \cdot \|t_2\|}.
$$

Then the exterior angle $\theta$ is $\pi - \alpha$.

First let’s do some computations when the boundary is smooth.

3) Verify the Gauss-Bonnet formula for the flat disk whose boundary is a circle of radius 1.

4) Verify the Gauss-Bonnet formula for the hemisphere whose boundary is a circle of radius 1.

5) Verify the Gauss-Bonnet formula for a cylinder of height 1 and radius 1.

6) Suppose that $y = h(x)$ for $a \leq x \leq b$ is a smooth curve which does not meet the $x$-axis. Consider the surface $S$ obtained by revolving around the $x$-axis:

$$
f(u,v) = (u, h(u) \cos(v), h(u) \sin(v))
$$

for $0 \leq v \leq 2\pi$ and $a \leq u \leq b$. Verify the Gauss-Bonnet formula for this surface with boundary.

7) Verify the Gauss-Bonnet formula for a “polar cap” of the unit sphere where we look at the portion of the unit sphere whose $z$-coordinates are above a certain value. (In spherical coordinates, we are putting an upper bound on the angle $\theta$.)

Finally let’s do some computations when the boundary is only piecewise smooth.

8) Suppose that $S$ is a “wedge” in the unit sphere cut out by two great circles meeting at a point with angle $\alpha$. Verify the Gauss-Bonnet formula for $S$. 
9) Suppose that $S$ is a triangle in the unit sphere whose edges are all great circles (i.e. geodesics) and whose interior angles are $\alpha, \beta, \gamma$. Using Gauss-Bonnet, verify that

$$\text{area}(S) = \alpha + \beta + \gamma - \pi.$$ 

10) Consider the surface $S$ defined by a square region in a helicoid:

$$f(u, v) = (u \cos(v), u \sin(v), v)$$

for $0 \leq u \leq 1$ and $0 \leq v \leq \pi$. Verify the Gauss-Bonnet formula for $S$. 