Lecture 4: Gauss-Bonnet

This set of exercises will give you more practice with the Gauss-Bonnet formula.

Let's start by remembering how to define the curvature for a smooth oriented parametric surface $S \subset \mathbb{R}^3$. (Here oriented means we have a compatible choice of direction for the normal vectors at every point.) Suppose our parametrization is given by a function f:

$$f(u, v) = (f_1, f_2, f_3).$$

As in lecture, we denote by $f_u, f_v, f_{uu}, f_{uv}, f_{vv}$ the vectors obtained by taking the single/double derivatives of the components of f. Finally, we also consider the normal unit vector

$$\widehat{n} = \frac{f_u \times f_v}{\|f_u \times f_v\|}.$$

(To be precise we should ensure that the direction of $\widehat{n}$ agrees with our orientation.) Then the Gaussian curvature of S at a point p is given by the formula

$$K = \frac{\langle f_{uu}, \widehat{n}\rangle\langle f_{vv}, \widehat{n}\rangle - \langle f_{uv}, \widehat{n}\rangle^2}{\langle f_u, f_u\rangle\langle f_v, f_v\rangle - \langle f_u, f_v\rangle^2}$$

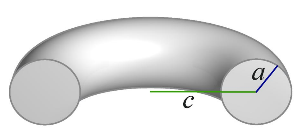
Now suppose that $S \subset \mathbb{R}^3$ is a smooth compact oriented surface.

Theorem 0.1 (Gauss-Bonnet)**.** *Let S be a smooth compact orientable surface with no boundary. Then*

$$\int_S K \, dA = 2\pi\chi(S)$$

where dA is the area element on S.

Let's try some examples!



Cross section: major radius c, minor radius a

First consider the torus parametrized by

$$f(u, v) = ((c + a\cos(v))\cos(u), (c + a\cos(v))\sin(u), a\sin(v))$$

for the range $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$. (Here u represents an angle in the x, y-plane and v represents the angle in the "small circle" formed by taking a cross-section of the torus.)

1) Verify carefully that the curvature satisfies

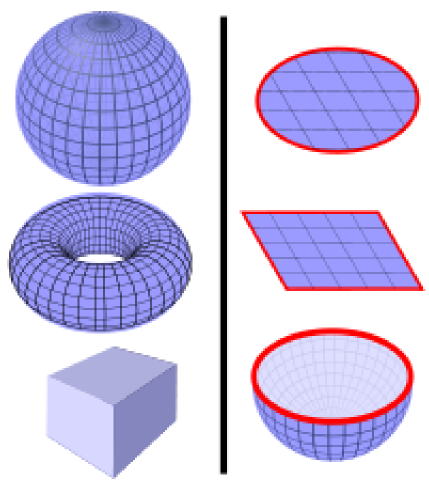
$$K(u,v) = \frac{\cos(v)}{a(c + a\cos(v))}$$

2) Verify that the Gauss-Bonnet formula holds for the parametrized torus above.

Unfortunately it becomes much more difficult to parametrize surfaces of genus ≥ 2. Instead we will go in a slightly different direction.

1 Gauss-Bonnet for surfaces with boundary

An important feature of the original Gauss-Bonnet formula is that it is stated for surfaces "without boundary": every point has an open neighborhood that is homeomorphic to an open disk in $\mathbb{R}^2$. In a surface "with boundary", every point has an open neighborhood that is homeomorphic to an open ball in the closed upper half-plane in $\mathbb{R}^2$.



Surfaces without boundary / with boundary

There is also a version of the Gauss-Bonnet formula that holds for surfaces with boundary. In this situation we will need to include a correction term to account for the removed region. This correction term comes from a quantity known as "geodesic curvature." Let's suppose the boundary curve is parametrized by a function g:

$$g(t) = (g_1, g_2, g_3).$$

We need to insist that the orientation of the parametrization is compatible with our surface orientation. This is determined by the right hand rule: suppose we choose a point

p in the boundary. Orient the thumb of the right hand along the chosen orientation for points near p. Then the fingers of the right hand will "curl around" the boundary in the direction of the orientation.

Given a parametrized curve g as above, the geodesic curvature is defined as

$$k_g = \frac{1}{\|g'(t)\|^2} g''(t) \cdot (\widehat{n} \times \widehat{t})$$

where $\widehat{n}$ is the unit normal vector to the surface and $\widehat{t}$ is the unit tangent vector to the curve.

Remark 1.1. A curve in S is said to be a geodesic if $k_g = 0$. Such curves are the "shortest paths" between two points in the surface S!

Example 1.2. Consider the circle $(\cos(t), \sin(t), 0)$ in the x, y-plane. We can think of this as the boundary of a flat region in the x, y-plane. We will orient this surface in the upward direction, so the unit normal is the constant vector $(0, 0, 1)$. (Note that our choice of orientation of the surface is compatible with our choice of orientation of the boundary via the right hand rule.) Therefore $\widehat{n} \times \widehat{t}$ is the vector $(0, 0, 1) \times (-\sin(t), \cos(t), 0) = (-\cos(t), -\sin(t), 0)$. Then the geodesic curvature is

$$k_g = \frac{1}{\sin(t)^2 + \cos(t)^2} (-\cos(t), -\sin(t), 0) \cdot (-\cos(t), -\sin(t), 0)$$

The various terms cancel out to give $k_g = 1$.

Example 1.3. Consider the circle $(\cos(t), \sin(t), 0)$ in the x, y-plane. This time we think of the circle as the boundary of the "spherical cap" lying over the circle. We will orient this surface in the upward direction; this means that the unit surface normal along the circle is given by $(\cos(t), \sin(t), 0)$. (Note that our choice of orientation of the surface is compatible with our choice of orientation of the boundary via the right hand rule.) Therefore $\widehat{n} \times \widehat{t}$ is the vector $(\cos(t), \sin(t), 0) \times (-\sin(t), \cos(t), 0) = (0, 0, 1)$. Then the geodesic curvature is

$$k_g = \frac{1}{\sin(t)^2 + \cos(t)^2} (-\sin(t), -\cos(t), 0) \cdot (0, 0, 1)$$

In this case we have $k_g = 0$.

We can now state a version of Gauss-Bonnet for surfaces with piecewise smooth boundary:

Theorem 1.4 (Gauss-Bonnet). *Let S be a smooth compact orientable surface with a piecewise smooth boundary which has corners at $\{v_i\}_{i=1}^r$. Then*

$$\int_S K\,dA + \int_{\partial S} k_g\,ds + \sum_{i=1}^r \theta_{v_i} = 2\pi\chi(S)$$

where dA is the area element on S, ds is the line element along the boundary ∂S, and θ_{v_i} denotes the exterior angle at v_i.

Remember, to compute the *interior* angle α between two smooth curves meeting at a point p, one computes their tangent vectors t_1, t_2 at p and computes the angle between them:

$$\cos(\alpha) = \frac{t_1 \cdot t_2}{\|t_1\| \cdot \|t_2\|}.$$

Then the *exterior* angle θ is $\pi - \alpha$.

First let's do some computations when the boundary is smooth.

3) Verify the Gauss-Bonnet formula for the flat disk whose boundary is a circle of radius 1.

4) Verify the Gauss-Bonnet formula for the hemisphere whose boundary is a circle of radius 1.

5) Verify the Gauss-Bonnet formula for a cylinder of height 1 and radius 1.

6) Suppose that $y = h(x)$ for $a \leq x \leq b$ is a smooth curve which does not meet the x-axis. Consider the surface S obtained by revolving around the x-axis:

$$f(u, v) = (u, h(u)\cos(v), h(u)\sin(v))$$

for $0 \leq v \leq 2\pi$ and $a \leq u \leq b$. Verify the Gauss-Bonnet formula for this surface with boundary.

7) Verify the Gauss-Bonnet formula for a "polar cap" of the unit sphere where we look at the portion of the unit sphere whose z-coordinates are above a certain value. (In spherical coordinates, we are putting an upper bound on the angle θ.)

Finally let's do some computations when the boundary is only piecewise smooth.

8) Suppose that S is a "wedge" in the unit sphere cut out by two great circles meeting at a point with angle α. Verify the Gauss-Bonnet formula for S.

9) Suppose that S is a triangle in the unit sphere whose edges are all great circles (i.e. geodesics) and whose interior angles are α, β, γ. Using Gauss-Bonnet, verify that

$$\text{area}(S) = \alpha + \beta + \gamma - \pi.$$

10) Consider the surface S defined by a square region in a helicoid:

$$f(u, v) = (u\cos(v), u\sin(v), v)$$

for $0 \leq u \leq 1$ and $0 \leq v \leq \pi$. Verify the Gauss-Bonnet formula for S.