Lecture 5: Plane curves

Suppose that \( C \subset \mathbb{R}^2 \) is a curve defined by a polynomial equation \( P(x, y) = 0 \). In lecture we mentioned that one can “compactify” \( C \) by taking its closure in \( \mathbb{P}^2_\mathbb{R} \). In this set of exercises we will discuss this operation in more detail. First a reminder:

**Definition 0.1.** A polynomial is said to be homogeneous if every term has the same degree.

For example, the polynomial \( x^2 + 2xy - y^2 \) is homogeneous of degree 2 while \( x^2 - 3y \) is not homogeneous.

### 1 Projective space

This section recalls some basic facts about projective space (also covered in Prof. Migliore’s lectures). We will think of \( \mathbb{P}^2_\mathbb{R} \) as the set of equivalence classes of triples \((x : y : z)\) of real numbers such that \( x, y, z \) are not all zero, where two triples \((x_1 : y_1 : z_1)\) and \((x_2 : y_2 : z_2)\) are said to be equivalent if there is a non-zero \( \lambda \in \mathbb{R} \) such that

\[
\begin{align*}
x_1 &= \lambda x_2 \\
y_1 &= \lambda y_2 \\
z_1 &= \lambda z_2
\end{align*}
\]

or expressed more compactly, \((x_1 : y_1 : z_1) = \lambda(x_2 : y_2 : z_2)\). We will somewhat lazily think of a non-zero triple \((x : y : z)\) as an element of \( \mathbb{P}^2 \), with the caveat that there are other triples that also define the same element.

We can identify a copy of \( \mathbb{R}^2 \) inside of \( \mathbb{P}^2 \) in the following way. Consider the function

\[
\phi : \mathbb{R}^2 \to \mathbb{P}^2 \\
\phi(x, y) = (x : y : 1)
\]

This function is injective: if \( \phi(x, y) = \phi(x', y') \) then \((x : y : 1)\) and \((x' : y' : 1)\) are equivalent tuples. But by looking at the \( z \)-coordinate we see the rescaling factor for these equivalent tuples must be 1, so that \( x = x' \) and \( y = y' \).

The image of this function is the set \( U \subset \mathbb{P}^2 \) consisting of all points \((x : y : z)\) such that \( z \neq 0 \). Indeed, if \((x : y : z)\) is a point with \( z \neq 0 \), then it is equivalent to the point \((\frac{x}{z} : \frac{y}{z} : 1)\) which is clearly in the image of \( \phi \). In other words, on \( U \) we have an inverse function

\[
\phi^{-1} : U \to \mathbb{R}^2 \\
\phi^{-1}(x : y : z) = (\frac{x}{z}, \frac{y}{z})
\]
We will think of $U$ as a “copy” of $\mathbb{R}^2$ inside of $\mathbb{P}^2$. We will call the complement $\mathbb{P}^2 \setminus U$ the “set of points at infinity” and denote it by $\mathbb{P}^1_{\infty}$. To be clear: the set of points at infinity is the set of points of the form $(x : y : 0)$ inside of $\mathbb{P}^2$ (which really is a copy of $\mathbb{P}^1$). Geometrically the point $(x : y : 0)$ represents the “slope direction” $\frac{y}{x}$.

Note that a polynomial $P(x, y, z)$ usually does not define a function $\mathbb{P}^2 \to \mathbb{R}$ in any meaningful way. Of course if we choose a particular triple $(x : y : z)$ then $P(x, y, z)$ is a well-defined real number. However, if we choose a rescaling $(\lambda x : \lambda y : \lambda z)$ representing the same point of $\mathbb{P}^2$, usually the value of $P(\lambda x, \lambda y, \lambda z)$ is different. Since the value of $P(x, y, z)$ depends on the choice of representative, we can’t use $P$ to define a function $\mathbb{P}^2 \to \mathbb{R}$.

However, we can do something a little weaker if $P(x, y, z)$ is a homogeneous polynomial. In this case, the rescaling operation is easier to understand: if $P$ has degree $d$ then

$$P(\lambda x, \lambda y, \lambda z) = \lambda^d P(x, y, z).$$

We still can’t necessarily get a value of $P$ at a point $(x : y : z) \in \mathbb{P}^2$. However, we can tell if $P(x, y, z) = 0$ because this property does not change when we rescale the point!

**Definition 1.1.** Given a homogeneous polynomial $\tilde{P}(x, y, z)$ of degree $d \geq 1$, the curve $\tilde{C} \subset \mathbb{P}^2$ defined by the equation $\tilde{P}(x, y, z) = 0$ is the set of all points $(x : y : z)$ such that $\tilde{P}(x, y, z) = 0$. (For emphasis: to check if a point of $\mathbb{P}^2$ is in $\tilde{C}$ it does not matter which representative tuple we pick!)

**2 Affine to projective**

Now suppose we have a plane curve $C$ defined by an equation $P(x, y) = 0$ in $\mathbb{R}^2$. We will associate to it a curve $\tilde{C} \subset \mathbb{P}^2$ in the following way.

**Definition 2.1.** Suppose that $P(x, y)$ is a degree $d$ polynomial with $d \geq 1$. We define the homogenization $\tilde{P}(x, y, z)$ to be the homogeneous polynomial of degree $d$ obtained by adding as many factors of $z$ as necessary to the terms of $P$ to obtain a homogeneous polynomial of degree $d$.

For example,

$$P(x, y) = xy^2 - 3x^2 + 2y \implies \tilde{P}(x, y, z) = xy^2 - 3x^2 z + 2yz^2$$

$$P(x, y) = x^4 - xy + 3 \implies \tilde{P}(x, y, z) = x^4 - xyz^2 + 3z^4$$

Note that it is possible that $\tilde{P}(x, y, z) = P(x, y)$ if $P(x, y)$ is already homogeneous.

**Definition 2.2.** Suppose that $C \subset \mathbb{R}^2$ is a curve defined by $P(x, y) = 0$ where $P(x, y)$ is a degree $d$ polynomial with $d \geq 1$. We define $\tilde{C} \subset \mathbb{P}^2$ by the equation $\tilde{P}(x, y, z) = 0$. 
Using the inclusion $\phi : \mathbb{R}^2 \to \mathbb{P}^2$ we can think of $C$ as a subset of $\mathbb{P}^2$. Loosely speaking, $\tilde{C}$ is the “closure” of $C$ inside of $\mathbb{P}^2$. The following theorem gives us some indication of why this is a valid perspective.

**Theorem 2.3.** For any curve $C \subset \mathbb{R}^2$ defined by a polynomial $P(x,y) = 0$ we have

$$\tilde{C} \cap U = \phi(C).$$

**Proof.** A point in $U$ can be rescaled to have the form $(x : y : 1)$. Note that $	ilde{P}(x, y, 1) = P(x, y)$. Thus the set of points of the form $(x : y : 1)$ such that $	ilde{P}(x, y, 1) = 0$ is the same as the set of points $(x, y)$ such that $P(x, y) = 0$. \qed

**Caution 2.4.** Strictly speaking it is not true that $\tilde{C}$ is the closure of $C$ in $\mathbb{P}^2$. When we are working over $\mathbb{C}$ this is literally true! But when we are working over $\mathbb{R}$ it is not; see Exercise (4). We will ignore this minor issue.

The following exercises give you practice with this “closure” operation. Feel free to use Desmos to get a visual intuition!

1) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $x^2 - y = 0$. What does the curve $\tilde{C}$ look like? What are its points at infinity?

2) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $6x^2 - y^2 - 5y = 0$. What does the curve $\tilde{C}$ look like? What are its points at infinity?

3) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $2x^2 + y^2 - 6 = 0$. What does the curve $\tilde{C}$ look like? What are its points at infinity?

4) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $x^2 + 1 = 0$. What does the curve $\tilde{C}$ look like? What are its points at infinity?

5) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $x^3 - 3y^3 + 3x - 5y = 0$. What does the curve $\tilde{C}$ look like? What are its points at infinity?

6) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $x^3 - 3x^2y + 3xy^2 - y^3 - 6x^2 + 5y^2 - 3x + y + 5 = 0$. What does the curve $\tilde{C}$ look like? What are its points at infinity?

7) Consider the curve $C \subset \mathbb{R}^2$ defined by the equation $x^3 - 4x^2y + 5xy^2 - 2y^3 - 5xy + 3x - 2 = 0$. What does the curve $\tilde{C}$ look like? What are its points at infinity?

8) Make sense of the following claims. Suppose $P(x, y)$ has degree $d \geq 1$. 

• Let \( P_d(x, y) \) denote all the terms of \( P(x, y) \) with degree \( d \). Then the points at infinity of \( \tilde{C} \) is the subset of \( \mathbb{P}_1^1 \) defined by the homogeneous polynomial \( P_d(x, y) = 0 \).

• The points at infinity of \( \tilde{C} \) are the “limits” of the tangent directions to the points of \( C \).

9) If \( \tilde{C} \subset \mathbb{P}_2^2 \) is defined by a homogeneous polynomial \( \tilde{P}(x, y, z) = 0 \), is it true that \( \tilde{C} \cap U \) is the same as the subset defined by the “dehomogenized” polynomial \( \tilde{P}(x, y, 1) = 0 \)? Prove or give a counterexample!

3 Smooth points

Let \( C \subset \mathbb{R}_2^2 \) be a curve defined by a polynomial \( P(x, y) = 0 \). Recall that \( p \) is said to be a singular point of \( C \) if both \( \frac{\partial P}{\partial x}(p) \) and \( \frac{\partial P}{\partial y}(p) \) vanish at \( p \). Otherwise \( p \) is said to be a smooth point of \( C \). (Technically, it is best to look for singular points in \( \mathbb{C}_2^2 \) and not just \( \mathbb{R}_2^2 \) – this will give us the best sense of the behavior of our curve.)

Caution 3.1. A singular point \( p \) of \( C \) must satisfy three conditions, not two: \( P(p) = 0, \frac{\partial P}{\partial x}(p) = 0, \frac{\partial P}{\partial y}(p) = 0 \)

10) Show that \((0, 0)\) is a singular point of \( C \) if and only every term of \( P \) has degree \( \geq 2 \).

Show that this matches your intuition by using Desmos to graph the following curves: \( y^2 - x^3 - x^2 = 0, (x^2 + y^2)^2 + 3x^2y - y^3 = 0 \).

11) Show that the following curves are smooth:
   a) \( x^2 + y^2 - 3 = 0 \).
   b) \( x^2 - 3y^2 + 5x - 6y = 0 \).
   c) \( y^2 - x^3 - 3x - 1 = 0 \).

12) Find all the singular points of the following curves:
    a) \( y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy = 0 \).
    b) \( x^4 + y^4 - x^2y^2 = 0 \).
    c) \( x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1 = 0 \).

13) Suppose that \( C \) has the Weierstrass form \( y^2 = x^3 + ax + b \). Show that \( C \) is singular if and only if \( 4a^3 + 27b^2 = 0 \). (Here I am implicitly looking for singular points over \( \mathbb{C} \).)
14) Suppose that $C$ is defined by a degree 2 equation $ax^2 + bxy + cy^2 + dx + ey + f = 0$. 
What is the condition on the coefficients that determines whether $C$ is smooth or has a singular point? (This is a little complicated – the analogous question in $\mathbb{P}^2$ is better behaved.)

Now suppose we have $\tilde{C} \subset \mathbb{P}^2$ defined by a polynomial equation $\tilde{P}(x, y, z) = 0$. Letting $U = \{(x : y : z)|z \neq 0\}$ as before, recall that $\tilde{C} \cap U$ is the same as the locus $C \subset \mathbb{R}^2$ defined by the equation $\tilde{P}(x, y, 1) = 0$. Thus it is very natural to say that a point in $\tilde{C} \cap U$ is singular or smooth if the corresponding point of $C$ is singular/smooth.

However, from the viewpoint of projective space, the points at infinity are as good as any other point. So we can also define a notion of singular/smooth for these points as well! One way is simply to swap the roles of the variables and “dehomogenize” with respect to $x$ or $y$:

- Letting $V = \{(x : y : z)|y \neq 0\}$, we check for singularities of $\tilde{C} \cap V$ using the equation $\tilde{P}(x, 1, z) = 0$.
- Letting $W = \{(x : y : z)|x \neq 0\}$, we check for singularities of $\tilde{C} \cap W$ using the equation $\tilde{P}(1, y, z) = 0$.

This looks a little complicated. For example, it is not at all obvious that if a point $p \in U \cap V \cap W$ is singular when considered in $\tilde{C} \cap U$, it is also singular when considered in $\tilde{C} \cap V$ or $\tilde{C} \cap W$. (To be clear: it is true, just not obvious.)

An easier option is given by:

**Definition 3.2 (Projective criterion).** Suppose $\tilde{C} \subset \mathbb{P}^2$ is defined by a homogeneous equation $\tilde{P}(x, y, z) = 0$. We say that $p \in \tilde{C}$ is a singular point if

$$\frac{\partial \tilde{P}}{\partial x}(p) = 0 \quad \frac{\partial \tilde{P}}{\partial y}(p) = 0 \quad \frac{\partial \tilde{P}}{\partial z}(p) = 0$$

Otherwise we say that $p$ is a smooth point.

15) Find all the singular points of the following curves:

a) $xz^2 - y^3 + xy^2 = 0$

b) $x^2y^2 + 36xz^3 + 24yz^3 + 108z^4 = 0$.

16) What condition on $k$ determines whether the curve $$(x + y + z)^3 - kxyz = 0$$ has singular points?
17) Suppose that \( C \subset \mathbb{R}^2 \) is defined by the equation \( y^2 = x^3 + ax + b \). Show that all points at infinity of the corresponding curve \( \tilde{C} \) are smooth.

18) Suppose that \( C \) is defined by a degree 2 equation
\[
ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0.
\]
What is the condition on the coefficients that determines whether \( C \) is smooth or has a singular point? (Hint: recall that every two lines in \( \mathbb{P}^2 \) intersect!)

19) Suppose that \( \tilde{P}(x, y, z) \) is homogeneous of degree \( d \). Prove Euler’s formula:
\[
d \cdot \tilde{P} = x \frac{\partial \tilde{P}}{\partial x} + y \frac{\partial \tilde{P}}{\partial y} + z \frac{\partial \tilde{P}}{\partial z}.
\]
Using this formula, prove that our two definitions of singular/smooth – one via “dehomogenizing”, one via the projective criterion – are equivalent. In particular, verify that if \( p \in \tilde{C} \) is singular then it is singular in every “dehomogenization” such that the corresponding copy of \( \mathbb{R}^2 \) contains \( p \).