Did Kurt Gödel Kill the Modern Mathematician?

An Analysis of the Impact of the Incompleteness Theorems on Modern Mathematics

By Luke Reifenberg

This paper will treat the philosophical significance of Gödel's incompleteness theorems in so much as they have impacted "daily [modern] mathematical work" (Feferman, 2006). Gödel's two incompleteness theorems have been labeled as "the most mathematically significant achievement of the 20th century," so it would seem an interesting undertaking to examine exactly how the theorems have been significant for the "working mathematician" (Feferman). [1] The examination will proceed in three parts: first, the explanations of terminology necessary for understanding the meaning of Gödel's theorems, namely, an explanation of the terms *formal system*, *completeness*, and *consistency*; second, a description of what Gödel's incompleteness theorems say about the completeness and consistency of a formal system; and third, an examination concerning why a working mathematician might care why a formal system is complete and/or consistent, augmented by an analysis of how Gödel's incompleteness theorems may impact these cares.

To begin, an explanation of terminology. Imprecisely, a *formal system* is a system of finite axioms furnished with effective rules of inference enabling one to prove new theorems. A formal system is *complete* (i.e. possesses *completeness*) if every meaningful and true statement in the system's formal language can be proved in the system, A formal system is *consistent* (i.e.

[1] The epithet "working mathematician" is borrowed from Franzén, 2006. The term, rather than making some sort of contrast with the "unemployed mathematician," is rather meant as a representative device to illustrate the perspective of the conventional modern mathematician.

possesses *consistency*) if there is no statement such that the affirmation and the negation of the statement are both provable in the system.

Gödel's first incompleteness theorem states that "any consistent formal system *F* within which a certain amount of elementary arithmetic can be carried out is incomplete" (Raatikainen, 2018). Gödel's second incompleteness theorem states that "for any consistent system *F* within which a certain amount of elementary arithmetic can be carried out, the consistency of *F* cannot be proved in *F* itself" (Raatikainen). The theorems originally applied to "extents of a variant P of the system of *Principia Mathematica*," but the results have been shown to hold more generally for all systems, excepting those "cooked up" systems which have some trivial way of being complete and consistent (such "exceptionally simple" systems incapable of "carrying out a certain amount of elementary arithmetic" may be of interest to logicians, but they are not the systems within which working mathematicians busy themselves) (Franzén, Feferman).

Let us now imagine ourselves working mathematicians and examine the significance of completeness and consistency of the arbitrary formal system we are doing work in, which we may term "*S*." For reasons that will soon be made clear, let us assume that *S* is consistent and that *S* can carry out a "certain amount of elementary arithmetic" (because these are the systems we care about and also the ones wherein Gödel's theorems apply) (Feferman). Two topics become immediately obvious to consideration: whether *S* is complete (and whether there exists a proof of the completeness) and whether *S* is consistent (and whether there exists a proof of the consistency).

As regards the completeness of *S*, one may at first think that being able to prove every true statement would be a nice thing to have, for if *S* was incomplete (which Gödel's first incompleteness theorem proves to be the case) we might be bothered by two difficulties: first, we may fear that we are plugging away at the fool's errand of proving an undecidable statement, and second, we may be philosophically disheartened that there are some truths out there that we can never prove, potentially robbing us of what Hilbert calls the "conviction of the solvability of every mathematical problem [which] is a powerful incentive to the worker" (Franzén). But we may not need to be troubled by these worries, as "mathematicians who work in set theory or areas closely connected with set theory have learned to recognize the kind of problem or conjecture that may well be affected by the incompleteness theorem," and furthermore "no famous arithmetical conjecture has been shown to be undecidable in ZFC" (Franzén). We need not worry that our attempts to prove a statement may be like waiting for an infinitely looping program to terminate, for not only can we detect (by intuition, according to Franzén) when our attempts to prove a conjecture are in vain, but moreover our experiences tell us that no conjecture is in vain,[2] and moreover there are always other conjectures to work away at given the "phenomenon of the inexhaustibility of mathematics" alleged by Gödel (Feferman).

As regards proving the completeness of *S*, Gödel's first theorem shows that obtaining such a proof is not possible, and if we were to find such a proof then we would need to reassess the consistency of *S*.

As regards the consistency of *S*, this is a matter of great importance, as consistency is a very nice thing to have.[3] We assume that "mathematics as it stands today is consistent," which must be assumed and not proven, for such a proof would seem to be at best meaningless and at

[2] This may not be convincing, as without a proof that there are no unprovable arithmetic conjectures the fact we have not found an unprovable arithmetic conjecture does not mean such conjectures do not exist. If we were to find an unprovable arithmetic conjecture (by means of finding proof of unprovability, which need not exist) then "the search for new axioms in mathematics would take on a new urgency." (Franzén) However, no such conjecture has been found, and so mathematicians have no explicit cause to worry in that respect.

[3] This is especially so for PA, for its inconsistency would mean that arithmetical contradictions could be derived; we would be in a bad spot indeed if one could derive $\neg(0=0)$.

worst disastrous (Franzén). Meaningless because it is an "elementary fact of logic" that an inconsistent system can prove any statement (by the principle of explosion), and so a proof of consistency (within an inconsistent system) is trivial (Raatikainen). Disastrous because, by Gödel's second incompleteness theorem which states that any consistent system cannot prove its own consistency, finding a proof of consistency would mean, by modus tollens, that the system is inconsistent. We must be satisfied with no proof of consistency, and we may comfort ourselves knowing that "the significance of consistency proofs as a means of justifying our mathematical reasoning is easily overstated" (Franzén). Furthermore, the consistency of *S* can often be proven in some system stronger than *S*.[4] However, wanting to prove the consistency of a system within itself is "not at all what mathematician normally seek to prove," and the internal unprovability of the consistency of a system retells an "insight, familiar since antiquity" that not all things can be justified ad infinitum but rather at some point you must begin somewhere, with "basic principles in our mathematical reasoning, principles that we can justify only in informal terms," and mathematicians acknowledge this. They "tend to be content with accepting that consistency of the most powerful formal theory to which they ordinarily refer in foundational contexts cannot be proven in ordinary mathematics [although it may be proven in informal terms]" (Franzén).

To the working mathematician, Gödel's theorems seem an interesting discovery, but they are by no means world-shattering (or math-shattering), as they "play no role in daily mathematical work" (Feferman). And although Gödel's theorems are "usually viewed, and rightly, as casting a pall over Hilbert's project that is not likely to be lifted," it seems improper to

[4] Let us call the stronger system S_2. One notes that S_2 cannot internally prove the consistency of S_2. An even stronger system, S_3, *could* prove the consistency of S_2, but here the problem simply keeps going in an infinite regression.

conclude that the objectivity of "mathematics, the supreme bastion of reason" has been cut down by Gödel's theorems (George and Velleman, 2002). To make such conclusions is to preach a eulogy over an empty coffin. Gödel's theorems, despite their "paramount relevance to logic," are "clearly irrelevant to the concerns of [the] working mathematician," who, very much alive, continues to plug away at mathematics (Feferman).

Works Cited

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