Math 60850, Final Exam
Dec 6, 2017. Due Monday Dec 11, 2017 at 10:00 am

Instructor: Bei Hu **Name:**____

There is a total of 150 points on this exam. There is a total of 8 problems. ***Hand in your exam paper to either the instructor at 174 HURL or to Kathy Phillips at 153 HURL when completed.***

Rules: You may consult books and class notes, go to libraries and search online, use calculators and softwares. ***The work should be your own.*** **You cannot ask for help from anyone other than yourself. If you need clarification on a problem, consult your instructor.**

When you quote a theorem or a result, give the page number and identify it. ***e.g., (3) Theorem on page 256. you must show your work.*** **No credit will be given if no work is shown even if the answer is correct.**

1 (20 points). A pair of fair dice is tossed. Let X_1 be the value of first dice, X_2 be the value of second dice, and $X = X_1 + X_2$ be the total of two dice. Obviously $P(X = 2) = P(X = 12) = \frac{1}{6} \cdot \frac{1}{6} = \frac{1}{36}$.

(a) Find $P(X = k)$ for $2 \leq k \leq 12$.

(b) Find EX, $VarX$.

(c) Which of the three pairs $a), (X_1, X_2);\ b), (X, X_1);\ c), (X, X_2)$ are independent? Circle your answer.

$a)$ independent, not independent;

$b)$ independent, not independent;

$c)$ independent, not independent;

(d) Find conditional expectation $E(X_1 | X = 7)$.

2 (18 points). A coin with head probability $P(X = 1) = p$ and tail probability $P(X = 0) = q = 1 - p$. The coin in thrown repeatedly until exactly n heads are obtained. Let $Y =$ "total number of tosses". Obviously, $P(Y = n) = p^n$.

(a) Find $P(Y = k)$ for $k = n, n + 1, n + 2, n + 3, \cdots$.

(b) Find the generating function $G_Y(s)$.

(c) Find EY.

3 (18 points). Two factories A and B manufacture drones. Factory A produces 25% of the drones and factory B produces 75% of the drones. The defective rate for factory A is 1 in 100 and the defective rate for factory B is 4 in 100.

(a) What is the probability a randomly selected drone is defective?

(b) Given that a drone is defective, what is the probability that it is from factory A?

(c) A shipment of 100 drones is received. Let X be the total number of defective drones. Find $EX, VarX$.

4 (18 points). The random variable X takes non-negative integer values. Suppose $G(s)$ is the generating function of the random variable X.

(a) Find $E(X^4)$ in terms of the function G.

(b) Is $\dfrac{G(s/2)+G(s/3)}{G(1/2)+G(1/3)}$ a generating function? Justify your answer.

(c) Find $\sum_{n=0}^{\infty} s^n P(X>n)$ in terms of $G(s)$.

5 (20 points). Here is a simple random walk. Let $S_n = S_{n-1} + X_n$ if $1 \le S_{n-1} \le N-1$, where X_i are independent, and takes the value $-1, 1$. Assume

$$P(X_i = 1) = p,\ P(X_i = -1) = q, \quad p + q = 1.$$

We further assume that

(1) N is an absorbing barrier, i.e., the process stops if it reaches N.

(2) 0 is an reflecting boundary, i.e., if $S_{n-1} = 0$, then S_n is assigned to be 1.

Let J_k be the mean duration of the walk.

(a) Show that $J_N = 0,\ J_0 = 1 + J_1$.

(b) Find the equation for J_k.

(b) Suppose $p > q$. Find J_k.

6 (20 points). Let $\{X(t)\}$ be a simple birth-death process, i.e., the birth intensity $\lambda_n = n\lambda$ and the death intensity $\mu_n = n\mu$. Let $p_j(t) = P(X(t) = j)$.

(a) Find the differential equation system for $p_j(t)$ $(j = 0, 1, 2, 3, \cdots)$.

(show all your work)

(b) Find $EX(t)$ for all t, assuming that $X(0) = 1$.

(show all your work)

7 (18 points). Suppose that X_j ($j = 1, 2, 3, \cdots$) are independent, identically distributed continuous random variables with density $f(x)$, $f(x) = 0$ for $-\infty < x < 1$. Define $Y_n = (X_1 \cdot X_2 \cdots X_n)^{1/n}$. Please quote the theorems in the book, do not start from the beginning.

(a) Suppose that $E(\ln X_1) = \int_1^\infty f(x)\ln(x)dx < \infty$, show that Y_n converges in distribution.

(b) Find a condition on $f(x)$ so that Y_n converges almost surely.

8 (18 points). Let $\{W_t\}$ be the standard Wiener process ($W_0 = 0, EW_t = 0, EW_t^2 = t$)

(a) Let $r > 0, \sigma > 0$ be positive constants. Let X_t satisfy the geometric Brownian motion, i.e.,

$$\frac{dX_t}{X_t} = rdt + \sigma^2 dW_t.$$

Using Itô's formula, find

$$d(\ln |X_t|)$$

(show all your work)

(b) Let $Y_t = (1+t^2)\cos W_t$. Find dY_t.

(show all your work)