Ph. D. Qualifying Exam: Applied Probability. June 1st, 2017

Instructor: Bei Hu **Candidate:**____________________________________

There are 8 problems. Show all your work.

1. Toss a fair coin repeatedly until two consecutive heads appear. Let X be the total of all head count, find EX, $VarX$. Here are some sample events (1 = head, 0 = tail):

$$\{1,1\}, X = 1 + 1 = 2;$$
$$\{1,0,1,1\}, X = 1 + 0 + 1 + 1 = 3;$$
$$\{1,0,0,0,1,1\}, X = 1 + 0 + 0 + 0 + 1 + 1 = 3;$$
$$\{1,0,0,1,0,1,1\}, X = 1 + 0 + 0 + 1 + 0 + 1 + 1 = 4;$$

2. Suppose X and Y are independent continuous random variables with the same density function $f(x) = e^{-x}$, $x > 0$.

(a) Find the density function for $X + Y$;

(b) Find the joint density function for $X, X + Y$.

3. Three factories A, B and C manufacture zoogles. Factory A produces 20% of the zoogles and factory B produces 75% of the zoogles. The remaining 5% of the zoogles are from factory C. The defective rate for factory A is 1 in 50 and the defective rate for factory B is 1 in 20. The defective rate for factory C is 1 in 100.

(a) What is the probability a randomly selected zoogle is defective?

(b) Given that a zoogle is defective, what is the probability that it is from factory C?

(c) A shipment of 100 zoogles is received from a distribution center (which distribute zoogles from all factories). What is the probability that all of them are not defective?

4. Here is a simple random walk. Let $S_n = S_0 + \sum_{i=1}^{n} X_i$, where X_i are independent, and takes the value $-1, 1$. Assume

$$P(X_i = 1) = p,\ P(X_i = -1) = q, \quad p + q = 1, \quad p > q.$$

Suppose that the sites 0 and N are absorbing barriers. Let T_k be the time it takes to go from the site k to one of the absorbing barriers. Let $J_k = E(T_k)$. Obviously $J_0 = 0$ and $J_N = 0$.

(a) Find the equation for J_k.

(b) Find J_k.

5. Suppose $G(s)$ is the generating function of the random variable X. Suppose also that (a) the random variable X takes positive integer values, and (b) X_j ($j = 1, 2, 3, \cdots$) have the same distribution as X, and (c) $X, X_1, X_2, \cdots$ are all independent.

(a) Find $E\left(\frac{1}{X}\right)$ in terms of the function G (and possibly derivatives or integrals of G).

(b) Let $Y = X_1 + X_2 + X_3 + \cdots X_X$. Find $E(Y)$ in terms of the function G (and possibly derivatives or integrals of G).

6. (a) Suppose that $S_n = X_1 + X_2 + \cdots X_n$ is a Martingale with respect to $X_1, X_2, \cdots, X_n$, i.e.,

$$E(S_{n+1}|X_1, \cdots, X_n) = S_n.$$

Prove that $E(X_i X_j) = 0$ if $i \neq j$.

(b). For $x > 0$, find the limit (quote explicitly any theorem you use)

$$\lim_{n\to\infty} \; 2^{-n} \sum_{k:|k-\frac{1}{2}n|\leq\frac{1}{2}x\sqrt{n}} \binom{n}{m}$$

7. Let W_t be the standard Brownian motion with $W_0 = 0$, $EW_t = 0$ and $VarW_t = t$. Suppose that $0 < s < t$.

(a) Find $E(W_t|W_s)$ and $Var(W_t|W_s)$.

(b) Using Itô's formula, find $d(W_t)^2$.

(c) Find

$$\int_0^t W_t dW_t$$

8. Let W_t be the standard Brownian motion with $W_0 = 0$, $EW_t = 0$ and $VarW_t = t$.

(a) Prove that $\{W_{t_1}, W_{t_2}, \cdots, W_{t_N}\}$ $(t_1 < t_2 < \cdots < t_N)$ satisfies a multivariate normal distribution.

(b) Suppose that $0 < s < t$. Find $E(W_s|W_t)$.