Ph. D. Qualifying Exam: Applied Probability.   June, 2020

ACMS Applied Probability Qualifying exam committee.

Candidate:____

There are 8 problems. Each problem from problems 1 to 4 is worth 10 points. Each problem from problems 5 to 8 is worth 15 points. Show all your work.

1. (10 points). Toss a coin until you get two consecutive heads. This is a fair coin with $P(H) = P(T) = 1/2$.

(a) Let X be the total number of tosses. Then

$$\{X = 0\} = \{X = 1\} = \emptyset,$$
$$\{X = 2\} = \{HH\},\ \{X = 3\} = \{THH\},\ \{X = 4\} = \{TTHH, HTHH\}, \cdots,$$

For $n \geq 3$, find $P(X = n)$ in terms of $P(X = n-1)$ and $P(X = n-2)$.

Clearly $P(X = 0) = P(X = 1) = 0, P(X = 2) = \frac{1}{4}$,

$$\begin{aligned} P(X=n) &= P(X=n|1st\ \ H)P(1st\ \ H) + P(X=n|1st\ \ T)P(1st\ \ T) \\ &= P(X=n|1st\ \ H, 2nd\ \ T)P(1st\ \ H)P(2nd\ \ T) + P(X=n|1st\ \ T)P(1st\ \ T) \\ &= \frac{1}{4}P(X=n-2) + \frac{1}{2}P(X=n-1) \end{aligned}$$

(b) What is the expected total number of tosses?

$$\begin{aligned} EX &= 2P(X=2) + \sum_{n\geq 3} nP(X=n) \\ &= \frac{1}{2} + \sum_{n\geq 3} n\Big(\frac{1}{4}P(X=n-2) + \frac{1}{2}P(X=n-1)\Big) \\ &= \frac{1}{2} + \sum_{n\geq 3}\frac{n-2}{4}P(X=n-2) + \sum_{n\geq 3}\frac{2}{4}P(X=n-2) \\ &\qquad + \sum_{n\geq 3}\frac{n-1}{2}P(X=n-1) + \sum_{n\geq 3}\frac{1}{2}P(X=n-1) \\ &= \frac{1}{2} + \frac{1}{4}EX + \frac{2}{4} + \frac{1}{2}EX + \frac{1}{2} \ = \ \frac{3}{4}EX + \frac{3}{2}, \end{aligned}$$

so that $EX = 6$.

2. (10 points). Let D be the region of a unit disk. Assume the joint density $f_{X,Y}(x,y)$ is given by

$$f_{X,Y}(x,y) = \frac{1}{\pi} I_D(x,y),$$

where I_D is the indicator function of D.

(a) Find the marginal density $f_X(x)$ and $f_Y(y)$.

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f_{X,Y}(x,y)dy = \frac{2}{\pi}\sqrt{1-x^2}, \qquad x \in (-1,1).$$

Similarly,

$$f_Y(y) = \frac{2}{\pi}\sqrt{1-y^2}, \qquad y \in (-1,1)$$

(b) Are X, Y independent? Justify your answer.

Since $f_{X,Y}(x,y) \neq f_X(x) f_Y(y)$, they are NOT independent.

(c) Find the conditional density $f_{X|Y}(x|y)$.

Whenever $f_Y(y)$ is nonzero,

$$f_{X|Y}(x,y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{1}{2\sqrt{1-y^2}} \qquad \textbf{if } x^2+y^2<1$$

3. (10 points). Suppose X and Y are independent random variables with exponential distribution with density function $f(x) = \begin{cases} x^{-2}, & 1 < x < \infty \\ 0, & x \le 1 \end{cases}$. Find the probability density function for

(a) $\sqrt{X}$;

$$F_{\sqrt{X}}(x) = P(\sqrt{X} \le x) = P(X \le x^2) = \int_1^{x^2} t^{-2} dt = 1 - x^{-2}, \qquad x \ge 1.$$

Therefore

$$f_{\sqrt{X}}(x) = \frac{d}{dx}(1 - x^{-2}) = 2x^{-3}, \qquad x \ge 1.$$

(b) XY;

For $z > 1$,

$$\begin{aligned} F_{XY}(z) &= P(XY \le z) = \int_1^{\infty} P(XY \le z | Y = y) f_Y(y) dy \\ &= \int_1^{\infty} P\Big(X \le \frac{z}{y} \Big| Y = y\Big) f_Y(y) dy \\ &= \int_1^{\infty} \Big(\int_1^{z/y} f_X(x) dx\Big) f_Y(y) dy \\ &= \int_1^{z} \Big(\int_1^{z/y} x^{-2} dx\Big) y^{-2} dy \\ &= \int_1^{z} \Big(1 - \frac{y}{z}\Big) \frac{1}{y^2} dy \\ &= 1 - \frac{1}{z} - \frac{1}{z} \ln z. \end{aligned}$$

Therefore

$$f_{XY}(z) = \frac{d}{dz}\Big(1 - \frac{1}{z} - \frac{1}{z} \ln z\Big) = \frac{1}{z^2} \ln z, \qquad z > 1.$$

4. (10 points). Factory A produces 65 % of a special brand of chips with defective rate 6 %. Factory B produces the remaining 35 % of the same umbrellas with defective rate 5 %.

(a) What is the defective rate of a randomly selected chip of this brand of this product?

$$P(Def) = P(Def|A)P(A) + P(Def|B)P(B) = 6\% \cdot 65\% + 5\% \cdot 35\% = 5.65\%.$$

(b) Given that an chip of this brand is defective, what is the probability that it is from Factory A?

$$P(A|Def) = \frac{P(A \cap Def)}{P(Def)} = \frac{P(Def|A)P(A)}{P(Def)} = \frac{6\% \cdot 65\%}{5.65\%} = 69\%.$$

5. (15 points). Suppose $G(s) = se^{s-1}$ is the generating function of the random variable X. The random variable X takes non-negative integer values

(a) Find $P(X = 0)$, $P(X = 1)$, $E(X)$ and $E(X^2)$.

$$P(X = 0) = G(0) = 0,$$
$$P(X = 1) = G'(0) = e^{-1},$$
$$EX = G'(1) = 2,$$
$$EX^2 = G''(1) + G'(1) = 3 + 2 = 5.$$

(b) Let X_1, X_2 be independent, identically distributed random variables with the same mass function as X. Find

$$E\frac{1}{X_1 + X_2}$$

$$G_{X_1+X_2}(s) = G_{X_1}(s)G_{X_2}(s) = s^2e^{2s-2}.$$

Therefore

$$E\frac{1}{X_1 + X_2} = E\int_0^1 s^{X_1+X_2-1}ds = \int_0^1 s^{-1}G_{X_1+X_2}(s)ds = \int_0^1 se^{2s-2}ds = \frac{1}{4} + \frac{1}{4}e^{-2}.$$

6. (15 points). Suppose that $\{S_n\}$ is a sequence of random walk with $S_n = X_0 + X_1 + \cdots + X_n$, where $X_0 = 0$, $P(X_i = 1) = p$ and $P(X_i = -1) = q$ where $p + q = 1$, and $X_1, X_2, \cdots, X_n, \cdots$ are i.i.d,

(a) Prove that S_n is a Martingale if and only if $p = q = \frac{1}{2}$.

$$E(S_{n+1}|X_1, \cdots, X_n) = E(S_n + X_{n+1}|X_1, \cdots, X_n) = S_n + EX_{n+1} = S_n + p - q.$$

Thus S_n is a Martingale if and only if $p - q = 0$. Since $p + q = 1$, this is true if and only if $p = q = \frac{1}{2}$.

(b) Suppose that $p \neq q$. Find the constants c_n $(n = 1, 2, 3, \cdots)$ so that $M_n = S_n + c_n$ is a Martingale.

Following the above computation,

$$E(M_{n+1}|X_1, \cdots, X_n) = S_n + p - q + c_{n+1} = M_n + (p - q) + c_{n+1} - c_n.$$

Thus M_n is a Martingale if and only if $(p - q) + c_{n+1} - c_n = 0$, i.e.,

$$c_{n+1} = c_n - (p - q).$$

It follows that

$$c_n = c_0 - n(p - q),$$

where c_0 can be any number.

7. (15 points). Let $\{S_n\}$ be a homogenous Markov chain taking integer values with the transition probability

$$p_{ij} = \begin{cases} 9/10 & \text{if } j = i+1, \\ 1/10 & \text{if } j = i, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find all persistent states.

For any i,

$$\sum_n p_i i(n) = \sum_n \left(\frac{1}{10}\right)^n < \infty,$$

so that NO state are persistent.

(b) Find all states i and j such that $i \to j$, i.e., i communicates with j.

$$p_{ij}(m) > 0 \textbf{ if } j \geq i \textbf{ for some } m \geq 0.$$

So that $i \to j$ if $j \geq i$.

(c) Find all states i and j such that $i \leftrightarrow j$, i.e., i and j intercommunicate.

Base on (b), NO states other than itself intercommunicate.

8. (15 points). Let B_t denote the Standard Brownian Motion $EB_t = 0$ amd $VarB_t = t$.

(a) Let $Y_t = (B_t + 5t)^2$. Find dY_t.

By Simple Itô's formula (page 545, Theorem 4)

$$dY_t = \Big(2(B_t + 5t)5 + \frac{1}{2}2\Big)dt + 2(B_t + 5t)dB_t = (10B_t + 50t + 1)dt + (2B_t + 10t)dB_t.$$

(b) Suppose X_t is a geometric Brownian Motion representing an assert, i.e.,

$$\frac{dX_t}{X_t} = rdt + \sigma dB_t,$$

where the constant r represents an interest rate and the constant σ represents the volatility. Find

$$d\ln|X_t|.$$

By Itô's formula (page 545, Theorem 3)

$$d\ln|X_t| = \Big(\frac{rX_t}{X_t} + \frac{1}{2}\Big(-\frac{1}{X_t^2}\Big)(\sigma X_t)^3\Big)dt + \frac{\sigma X_t}{X_t}dB_t = (r - \frac{1}{2}\sigma^2)dt + \sigma dB_t.$$

(c) Following (b), find X_t

Integrating (b) over $[0, t]$**,**

$$\ln\frac{X_t}{X_0} = (r - \frac{1}{2}\sigma^2)t + \sigma B_t,$$

so that

$$X_t = X_0 \exp\Big((r - \frac{1}{2}\sigma^2)t + \sigma B_t\Big).$$

(d) Two investments (both satisfy geometric Brownian Motions) with investment B twice the interest rate and volatility of investment A, and the interest rate and volatility for investment A are $r = 0.05$ and $\sigma = 0.2$. Which one is better in average? Justify your answer.

A: $(r - \frac{1}{2}\sigma^2) = 0.05 - \frac{1}{2}(0.2)^2 = 0.03$

B: $(r - \frac{1}{2}\sigma^2) = 0.10 - \frac{1}{2}(0.4)^2 = 0.02$**.**

Therefore A **is better.**