Some notes on Shifted Wave Interference Fourier Transform Spectroscopy (SWIFTS)

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1 Basic analysis

Consider a fast detector placed at the output of a Michelson interferometer. The instantaneous power on the detector is proportional to

$$S(t,\tau) = \frac{1}{2}(E(t) + E(t-\tau))^2 \tag{1}$$

$$= \frac{1}{2}(E^2(t) + E^2(t-\tau) + 2E(t)E(t-\tau)) \tag{2}$$

where $E(t)$ is the field of the fixed arm and $E(t-\tau)$ is the field of the stage arm, delayed by τ. Assume that the fields can be expressed as a superposition of cavity modes as

$$E(t) = \sum_n E_n e^{i\omega_n t}, \tag{3}$$

where $E_n \equiv E(\omega_n)$ and the summation is taken over both positive n and negative n. Note that we do **not** initially assume that the cavity modes are equidistant. We assume only that $E_{-n} = E_n^*$ and $\omega_{-n} = -\omega_n$ since the field is real. The instantaneous power then reduces to

$$S(t,\tau) = \frac{1}{2} \sum_{n,m} E_n E_m e^{i(\omega_n+\omega_m)t} + E_n E_m e^{i(\omega_n+\omega_m)(t-\tau)} + 2E_n E_m e^{i(\omega_n+\omega_m)t - i\omega_m\tau} \tag{4}$$

$$= \frac{1}{2} \sum_{n,m} e^{i(\omega_n+\omega_m)t}(E_n E_m + E_n E_m e^{-i(\omega_n+\omega_m)\tau} + 2E_n E_m e^{-i\omega_m\tau}) \tag{5}$$

$$= \frac{1}{2} \sum_{n,m} E_n E_m^* e^{i(\omega_n-\omega_m)t}(1 + e^{-i(\omega_n-\omega_m)\tau} + 2e^{i\omega_m\tau}) \tag{6}$$

where in the last line we have replaced m with -m.

In conventional FTS, one records the signal $S_0(\tau) \equiv \langle S(t,\tau)\rangle$, where the integration is performed over lab timescales (Hz). In SWIFTS, one first demodulates the power with an *arbitrary* local oscillator of frequency ω_0, recording $S_+(\tau) \equiv \langle S(t,\tau)e^{-i\omega_0 t}\rangle$. (The LO can either be from a low-noise synthesizer [1], or it can be derived from the source itself in the so-called self-referenced scheme [2, 3].)

In conventional FTS, the signal is

$$S_0(\tau) = \frac{1}{2} \sum_{n,m} \langle E_n E_m^* e^{i(\omega_n-\omega_m)t}\rangle(1 + e^{-i(\omega_n-\omega_m)\tau} + 2e^{i\omega_m\tau}). \tag{7}$$

Due to the integration, the only terms that survive this summation are those terms which have $\omega_n = \omega_m$. Keeping only these terms, it simplifies to

$$S_0(\tau) \sim \frac{1}{2} \sum_{n} \langle E_n E_n^*\rangle \left(1 + 1 + 2e^{i\omega_n\tau}\right) \tag{8}$$

$$S_0(\tau) = \sum_{n} \langle |E_n|^2\rangle \left(\underbrace{\frac{1}{2}}_{\text{fixed}} + \underbrace{\frac{1}{2}}_{\text{variable}} + \underbrace{e^{i\omega_n\tau}}_{\text{interferometric}} \right). \tag{9}$$

As expected, there are three terms: two non-interferometric DC terms whose amplitude is half the total power of the source (corresponding to the fixed and variable arms of the interferometer), and an interferometric term whose amplitude is the power of the line.

In SWIFTS, the measured signal is

$$S_+(\tau) = \frac{1}{2} \sum_{n,m} \langle E_n E_m^* e^{i(\omega_n - \omega_m - \omega_0)t} \rangle (1 + e^{-i(\omega_n - \omega_m)\tau} + 2e^{i\omega_m \tau}). \tag{10}$$

Once again we have a double summation, but this time the only terms that will be measured are those that have $\omega_n - \omega_m = \omega_0$. For an arbitrary source (say from the modes of a dispersed cavity), practically all of these terms will vanish since most pairs of lines will have $\omega_n - \omega_m \neq \omega_0$. More explicitly, it must be the case that $|\omega_n - \omega_m - \omega_0|$ is less than the final integration bandwidth, which is usually on the order of Hertz. Without loss of generality, we may write

$$S_+(\tau) = \frac{1}{2} \sum_{n} \langle E(\omega_n + \omega_0) E^*(\omega_n) \rangle (1 + e^{-i\omega_0 \tau} + 2e^{i\omega_n \tau}) \tag{11}$$

$$= \sum_{n} \langle E(\omega_n + \omega_0) E^*(\omega_n) \rangle \Big(\underbrace{\frac{1}{2}}_{\text{fixed}} + \underbrace{\frac{1}{2} e^{-i\omega_0 \tau}}_{\text{variable}} + \underbrace{e^{i\omega_n \tau}}_{\text{interferometric}} \Big). \tag{12}$$

Again, this is a general result and does not rely on a comb structure. Now, for a dispersed source or a poorly-chosen local oscillator frequency, the product $\langle E(\omega_n + \omega_0) E^*(\omega_n) \rangle$ will obviously vanish since there will not simultaneously be a nonzero $E(\omega_n + \omega_0)$ and a nonzero $E(\omega_n)$.

Also, note that as in normal FTS there are three terms—the fixed arm, variable arm, and interferometric terms—but the variable arm term now oscillates at the local oscillator frequency rather than DC.

If we **finally** assume that the underlying structure of the structure of the source is an equidistant comb with repetition rate $\Delta\omega$ and that the local oscillator is matched to it (so that $\omega_0 = \Delta\omega$), then $E(\omega_n + \omega_0) = E_{n+1}$:

$$S_+(\tau) = \sum_{n} \langle E_{n+1} E_n^* \rangle (\frac{1}{2} + \frac{1}{2} e^{-i\omega_0 \tau} + e^{i\omega_n \tau}). \tag{13}$$

This implies that the phase of each nontrivial Fourier component is simply the phase difference of adjacent comb lines. Since group delay is just $\tau_g = \frac{\partial \phi}{\partial \omega} \sim \frac{\Delta \phi}{\Delta \omega}$, the phase of each SWIFTS line can therefore be interpreted as a group delay (modulo the repetition rate).

2 Alternate derivation

Alternatively, one can derive everything explicitly in terms of the quadrature interferograms:

$$S_I(\tau) \equiv \langle S(t,\tau)\cos\omega_0 t\rangle = \frac{1}{2}\langle S(t,\tau)(e^{i\omega_0 t} + e^{-i\omega_0 t})\rangle \tag{14}$$

$$S_Q(\tau) \equiv \langle S(t,\tau)\sin\omega_0 t\rangle = -\frac{i}{2}\langle S(t,\tau)(e^{i\omega_0 t} - e^{-i\omega_0 t})\rangle \tag{15}$$

Then

$$\begin{aligned} S_I(\tau) = \frac{1}{4} \; & \sum_{n,m}\langle E_n E_m^* (e^{i(\omega_n-\omega_m+\omega_0)t} + e^{i(\omega_n-\omega_m-\omega_0)t})\rangle\times \\ & \left(1 + e^{-i(\omega_n-\omega_m)\tau} + 2e^{i\omega_m\tau}\right) & (16) \\ S_Q(\tau) = -\frac{i}{4} \; & \sum_{n,m}\langle E_n E_m^* (e^{i(\omega_n-\omega_m+\omega_0)t} - e^{i(\omega_n-\omega_m-\omega_0)t})\rangle\times \\ & \left(1 + e^{-i(\omega_n-\omega_m)\tau} + 2e^{i\omega_m\tau}\right). & (17) \end{aligned}$$

The averaging will select only $\omega_n - \omega_m = -\omega_0$ in the first term and $\omega_n - \omega_m = \omega_0$ in the second,

$$\begin{aligned} S_I(\tau) = \frac{1}{4} \; & \sum_n \langle E(\omega_n-\omega_0)E^*(\omega_n)\rangle(1 + e^{i\omega_0\tau} + 2e^{i\omega_n\tau}) + \\ & \langle E(\omega_n+\omega_0)E^*(\omega_n)\rangle(1 + e^{-i\omega_0\tau} + 2e^{i\omega_n\tau}) & (18) \\ S_Q(\tau) = -\frac{i}{4} \; & \sum_n \langle E(\omega_n-\omega_0)E^*(\omega_n)\rangle(1 + e^{i\omega_0\tau} + 2e^{i\omega_n\tau}) - \\ & \langle E(\omega_n+\omega_0)E^*(\omega_n)\rangle(1 + e^{-i\omega_0\tau} + 2e^{i\omega_n\tau}) & (19) \end{aligned}$$

Keeping only the interferometric terms,

$$S_I(\tau) = \frac{1}{2} \sum_n \left(\langle E(\omega_n-\omega_0)E^*(\omega_n)\rangle + \langle E(\omega_n+\omega_0)E^*(\omega_n)\rangle\right) e^{i\omega_n\tau} \tag{20}$$

$$S_Q(\tau) = -\frac{i}{2} \sum_n \left(\langle E(\omega_n-\omega_0)E^*(\omega_n)\rangle - \langle E(\omega_n+\omega_0)E^*(\omega_n)\rangle\right) e^{i\omega_n\tau} \tag{21}$$

Of course, these can be added together to obtain

$$S_-(\tau) \equiv S_I(\tau) + iS_Q(\tau) = \sum_n \langle E(\omega_n - \omega_0) E^*(\omega_n)\rangle e^{i\omega_n \tau} \tag{22}$$

$$S_+(\tau) \equiv S_I(\tau) - iS_Q(\tau) = \sum_n \langle E(\omega_n + \omega_0) E^*(\omega_n)\rangle e^{i\omega_n \tau} \tag{23}$$

which is the previous result, since by definition $S_+(\tau) = \langle S(t,\tau)\cos(\omega_0 t)\rangle - i\,\langle S(t,\tau)\sin(\omega_0 t)\rangle = \langle S(t,\tau)e^{-i\omega_0 t}\rangle$.

3 Existence of non-zero S_Q

Examining the coefficients of S_Q,

$$S_Q = \frac{1}{2i}\sum_n \left(E_{n-1}E_n^* - E_{n+1}E_n^*\right) e^{i\omega_n\tau} \tag{24}$$

$$= \frac{1}{2i}\sum_{n>0} \left(E_{n-1}E_n^* - E_{n+1}E_n^*\right) e^{i\omega_n\tau} + \left(E_{-n-1}E_{-n}^* - E_{-n+1}E_{-n}^*\right) e^{i\omega_{-n}\tau} \tag{25}$$

$$= \frac{1}{2i}\sum_{n>0} \left(E_{n-1}E_n^* - E_{n+1}E_n^*\right) e^{i\omega_n\tau} + \left(E_{n+1}^*E_n - E_{n-1}^*E_n\right) e^{-i\omega_n\tau} \tag{26}$$

$$= \frac{1}{2i}\sum_{n>0} \left(E_{n-1}E_n^* - E_{n+1}E_n^*\right) e^{i\omega_n\tau} - \left(\left(E_{n-1}E_n^* - E_{n+1}E_n^*\right) e^{i\omega_n\tau}\right)^* \tag{27}$$

$$= \sum_{n>0} \mathrm{Im}\left(\left(E_{n-1}E_n^* - E_{n+1}E_n^*\right) e^{i\omega_n\tau}\right) \tag{28}$$

which proves that it is indeed real and shows that it is non-zero as long as there exists some n for which $E_{n-1} \neq E_{n+1}$ and $E_n \neq 0$.

4 Effect of non-equidistant lines

At first glance, it may appear that the lock-in time constant sets the bandwidth over which equidistance is assessed. However, this is incorrect—the lock-in bandwidth matters little, and the actual bandwidth is the inverse of the time it takes to acquire an interferogram. The Fourier transform does this for free. To see why, reconsider the time-average in Equation 10. Assume that it is in fact very weak (with a sub-μs time constant). In other words, its effect is only to filter out those beatnotes which have $|\omega_n - \omega_m - \omega_0| >> 0$, i.e. $n - m \neq 1$. Neglecting non-interferometric terms,

$$S_+(\tau) = \sum_{n,m} \langle E_n E_m^* e^{i(\omega_n - \omega_m - \omega_0)t} \rangle e^{i\omega_m \tau} \tag{29}$$

$$= \sum_n E_{n+1} E_n^* e^{i(\omega_{n+1} - \omega_n - \omega_0)t} e^{i\omega_n \tau} \equiv \sum_n E_{n+1} E_n^* e^{i\Delta\omega_n t} e^{i\omega_n \tau}. \tag{30}$$

This is effectively the same result, but now there is an explicit time-dependence in addition to a delay-dependence. Note, however, that this time-dependence has *no correlation with delay.* In step-scan mode, it is equivalent to a random phase modulation of $e^{i\omega_n \tau}$. In linear scan mode, delay becomes an explicit function of time, and $\tau = \frac{2v}{c} t$ where v is the speed of the moving arm and t is supported over $t \in [0, T]$, where T is the duration of the measurement. We then get

$$S_+(\tau) = \sum_n E_{n+1} E_n^* e^{i\Delta\omega_n \tau \frac{c}{2v}} e^{i\omega_n \tau} \tag{31}$$

$$= \sum_n E_{n+1} E_n^* e^{i(\omega_n + \Delta\omega_n \frac{c}{2v})\tau}. \tag{32}$$

In other words, the frequency that appears on the interferogram is no longer f_n, it is $f_n + \Delta f_n \frac{c}{2v}$ (a Doppler shift). Such peaks would appear at the wrong location when Fourier transformed. Since $c >> v$, even small error frequencies give rise to large errors for the peak's location. For example, at a speed of $v = 0.1$ mm/s, a pair of lines beating with a consistent error of $\Delta f_n =$10 Hz would have their corresponding frequency shifted by 15 THz. If they are beating incoherently, that peak will be *broadened* by 15 THz.

To see how this broadening would affect equidistance measurements, compare this broadening to the broadening of the peak induced by finite stage delay. The Fourier-limited broadening of the ω_n peak is just the maximum delay $c/(2L) = c/(2vT)$, which means that broadening becomes significant only when

$$\Delta f_n \frac{c}{2v} = \frac{c}{2Tv} \tag{33}$$

$$\Delta f_n = \frac{1}{T}, \tag{34}$$

proving the claim that it is the acquisition time that matters, not the lock-in time constant. Note that proper equidistance measurements require a deconvolution step where the effect of the interferometer's apodization function is removed, and this is where incoherence manifests. Attempting to fit a narrow peak to a broad peak by adjusting the amplitude will result in an amplitude that is far too small. This can be shown explicitly when the travel distance is chosen to coincide with an integer multiple of the repetition period, which causes the exponentials to be orthogonal. In this case,

$$S_0^{(n)} = \int E_n E_n^* d\tau \tag{35}$$

$$S_+^{(n)} = \int E_{n+1} E_n^* e^{i(\omega_{n+1,n}-\omega_0)\frac{c}{2v}\tau} d\tau. \tag{36}$$

The coherence is assessed in terms of the Fourier coefficients using

$$g(\omega_n) = \frac{|S_+^{(n)}|}{\sqrt{S_0^{(n)} S_0^{(n+1)}}}. \tag{37}$$

In the absence of amplitude noise,

$$g(\omega_n) = \left| \int e^{i(\phi_{n+1}-\phi_n)} e^{i(\omega_{n+1,n}-\omega_0)\frac{c}{2v}\tau} d\tau \right|. \tag{38}$$

When there are no equidistance errors and no phase noise, $g = 1$. If either are present, $g < 1$. If the maximum delay is given by $\tau_{max} = 2L/c$, then the lowest equidistance error that produces $g = 0$ occurs when $(\omega_{n+1,n} - \omega_0)\frac{c}{2v}\tau_{max} = 2\pi$, or

$$f_{n+1,n} - f_0 = \frac{v}{L} = \frac{1}{T} \tag{39}$$

where T is the total duration of the measurement. Thus, we have shown that the effective measurement bandwidth is $1/T$.

5 Equivalence to a Doppler shift

It is well-known that there is an interesting parallel between dual comb spectroscopy and FTIR if one considers that the moving mirror Doppler shifts the reflected frequencies [4]. Equation (30) can be recast to show this explicitly, by trading between the delay time and the lab time (using $\tau = \frac{2v}{c}t$ again)

$$S_+(\tau) = \sum_n E_{n+1}E_n^* e^{i(\omega_{n+1}-\omega_n-\omega_0)t} e^{i\omega_n \tau} \tag{40}$$

$$= \sum_n E_{n+1}E_n^* e^{i(\omega_{n+1}-\omega_n-\omega_0)t} e^{i\omega_n \frac{2v}{c}t} \tag{41}$$

$$= \sum_n E_{n+1}E_n^* e^{i(\omega_{n+1}-\omega_n(1-\frac{2v}{c})-\omega_0)t}. \tag{42}$$

Like all summations in these notes, this one is taken over both positive and negative components. At this point, we split it up:

$$S_+(\tau) = \sum_{n>0} E_{n+1}E_n^* e^{i(\omega_{n+1}-\omega_n(1-\frac{2v}{c})-\omega_0)t} + E_{-n+1}E_{-n}^* e^{i(\omega_{-n+1}-\omega_{-n}(1-\frac{2v}{c})-\omega_0)t} \tag{43}$$

$$= \sum_{n>0} E_{n+1}E_n^* e^{i(\omega_{n+1}-\omega_n(1-\frac{2v}{c})-\omega_0)t} + E_{n-1}^* E_n e^{i(-\omega_{n-1}+\omega_n(1-\frac{2v}{c})-\omega_0)t} \tag{44}$$

$$= \sum_{n>0} E_{n+1}E_n^* e^{i(\omega_{n+1}-\omega_n(1-\frac{2v}{c})-\omega_0)t} + E_n E_{n-1}^* e^{i(\omega_n(1-\frac{2v}{c})-\omega_{n-1}-\omega_0)t} \tag{45}$$

$$= \sum_{n>0} E_{n+1}E_n^* e^{i(\omega_{n+1}-\omega_n(1-\frac{2v}{c})-\omega_0)t} + E_{n+1}E_n^* e^{i(\omega_{n+1}(1-\frac{2v}{c})-\omega_n-\omega_0)t} \tag{46}$$

$$= \sum_{n>0} E_{n+1}E_n^* e^{i(\omega_{n+1}-\omega_n-\omega_0)t} \left(e^{i\omega_n \frac{2v}{c}t} + e^{-i\omega_{n+1}\frac{2v}{c}t} \right) \tag{47}$$

In other words, each beatnote is both Doppler-shifted up and Doppler-shifted down by nearly the same amount. Note, however, that Eqn. (47) is fully identical to (30), meaning that this has no effect on the conclusions reached in the previous section.

References

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