**Math 20580** Name:____________________

**Final Exam** Instructor:____________________

**May 7, 2015** Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 2 hours to do the test. You may leave earlier if you are finished.

There are 20 multiple choice questions worth 7 points each. You will receive 10 points for being present and following the instructions. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

**Sign the pledge.** "On my honor, I have neither given nor received unauthorized aid on this Exam":

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|---|---|---|---|---|---|---|---|---|---|---|---|
| 1. | a | b | c | d | e | 11. | a | b | c | d | e |
| 2. | a | b | c | d | e | 12. | a | b | c | d | e |
| 3. | a | b | c | d | e | 13. | a | b | c | d | e |
| 4. | a | b | c | d | e | 14. | a | b | c | d | e |
| 5. | a | b | c | d | e | 15. | a | b | c | d | e |
| 6. | a | b | c | d | e | 16. | a | b | c | d | e |
| 7. | a | b | c | d | e | 17. | a | b | c | d | e |
| 8. | a | b | c | d | e | 18. | a | b | c | d | e |
| 9. | a | b | c | d | e | 19. | a | b | c | d | e |
| 10. | a | b | c | d | e | 20. | a | b | c | d | e |

1. Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be a linear transformation such that

$$T\left(\begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -5 \end{bmatrix}, \quad T\left(\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}.$$

Which matrix below is the standard matrix for $T$?

(a) $\begin{bmatrix} 3 & 2 & 1 \\ 6 & 5 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} -2 & 3 & -3 \\ 5 & 6 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 2 & 2 & -3 \\ -5 & 4 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \end{bmatrix}$

(e) cannot be determined from the given information.

2. Consider the three differential equations

$$\text{(I) } y'' - y' - 2y = 0 \qquad \text{(II) } y'' + 2y' + 2y = 0 \qquad \text{(III) } y'' + 6y' + 9y = 0$$

Which of these equations admits a solution that satisfies $\lim_{t\to\infty} y(t) = \infty$?

(a) only (II) and (III) (b) only (I) and (II) (c) all three
(d) none (e) only (I)

3. Let $W = \operatorname{span}\left\{\begin{bmatrix}0\\-1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\1\end{bmatrix}\right\}$. Compute the projection of the vector $\begin{bmatrix}0\\1\\0\end{bmatrix}$ onto $W$.

(a) $\begin{bmatrix}-\frac{1}{3}\\ \frac{13}{15}\\ -\frac{1}{15}\end{bmatrix}$ (b) $\begin{bmatrix}-\frac{1}{15}\\ \frac{1}{5}\\ \frac{13}{15}\end{bmatrix}$ (c) $\begin{bmatrix}\frac{12}{15}\\ \frac{3}{5}\\ \frac{1}{15}\end{bmatrix}$ (d) $\begin{bmatrix}-\frac{1}{5}\\ -\frac{1}{15}\\ \frac{2}{5}\end{bmatrix}$ (e) $\begin{bmatrix}\frac{12}{15}\\ \frac{13}{15}\\ \frac{-1}{15}\end{bmatrix}$

4. One of the solutions of the equation

$$t^2y'' - 3ty' + 3y = 0, \qquad t > 0$$

is $y_1(t) = t$. Find a second solution $y_2$ which is not a scalar multiple of $y_1$.

(a) $t^2e^t$ (b) $t^2$ (c) $1/t$ (d) $t\cos t$ (e) $t^3$

5. Find a least squares solution of $Ax = b$ where $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 2 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

(a) $\begin{bmatrix} \dfrac{25}{21} \\ -\dfrac{10}{21} \end{bmatrix}$ (b) $\begin{bmatrix} 5 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} -\dfrac{15}{21} \\ \dfrac{5}{21} \end{bmatrix}$ (d) $\begin{bmatrix} \frac{2}{5} \\ \frac{1}{2} \end{bmatrix}$ (e) $\begin{bmatrix} \dfrac{21}{25} \\ -\dfrac{10}{25} \end{bmatrix}$

6. Use the method of variation of parameters to find the general solution of the differential equation

$$y'' + y = \frac{1}{\cos t}$$

(a) $y(t) = c_1 \cos t + c_2 \sin t + t \log|\cos t|$ (b) $y(t) = c_1 + c_2 \cos t \sin t$
(c) $y(t) = c_1 \cos t + c_2 \sin t + \log|\sin t|$ (d) $y(t) = c_1 \cos t + c_2 \sin t$
(e) $y(t) = c_1 \cos t + c_2 \sin t + \cos t \log|\cos t| + t \sin t$

7. Let $A$ be a $5 \times 6$ matrix. If the null space of $A^T$ has dimension 3 what is the dimension of the null space of $A$?

(a) 3 (b) 2 (c) 1 (d) 4 (e) 0

8. Consider the equation

$$t^2y'' + ty' + (t^2 - 5)y = 0, \qquad t > 0.$$

Find the Wronskian of the fundamental set of solutions of this equation determined by the conditions $y_1(1) = 2$, $y_2(1) = 0$, $y_1'(1) = -1$ and $y_2'(1) = 1$.

(a) $e^{t^3/3-5t}$ (b) $\dfrac{2}{t}$ (c) $2t$ (d) 2 (e) $e^{2t}$

9. Find the solution of the initial value problem

$$\begin{cases} y'' + 3y' + 2y = 0 \\ y(0) = 0, \quad y'(0) = -1 \end{cases}$$

(a) $e^{-2t} - e^{-t}$ (b) $e^{-t} - e^{2t}$ (c) $-te^{-t}$ (d) $e^{-3t} - e^{-2t}$ (e) $e^{t} - e^{-2t}$

10. The eigenvalues of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}$ are

(a) $1, -\sqrt{2}, \sqrt{2}$ (b) $1, -1, -1$ (c) $-\sqrt{2}, -1, \sqrt{2}$ (d) $0, 1, -1$ (e) $0, 1, -\sqrt{2}$

11. Classify the ordinary differential equation $y' + \cos y = 1$ as

(a) separable and linear
(b) exact and autonomous
(c) first order and linear
(d) exact and separable
(e) first order and autonomous

12. Which formula describes implicitly the solution of the initial value problem

$$\frac{dy}{dx} = \frac{x+1}{x \cdot (y^2+1)}, \quad y(1) = 0, \qquad x > 0.$$

(a) $y^3 + 3y = 3x + \log x^3 - 3$
(b) $x - \log x - y^3 = 1$
(c) $3x + \log x - 3y = 3$
(d) $-y^3 + 3y = 3x + 3\log x = 0$
(e) $y^3 + y = 3x - 3$

13. Suppose that $y_1$ is a solution of the homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

The function $y_2 = v(t) \cdot y_1(t)$ is a solution of the non-homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

if the following equality holds:

(a) $y_1 v'' + (2y_1' + py_1) \cdot v' = 0$ (b) $p'y_1' + q'y_2' = g$ (c) $g = 0$
(d) $v = e^{-\int p(t)dt}$ (e) $y_1 v'' + (2y_1' + py_1) \cdot v' = g$

14. Find the solution to the initial value problem

$$y' + 3t^2 y = 3t^2, \qquad y(-1) = 1 - e.$$

(a) $y = 1 + e^{-t^3}$ (b) $y = 1 - e^{t^3}$ (c) $y = 1 - e$
(d) $y = 1 - e^{-t^3}$ (e) $y = 1 + e^{t^3}$

15. If $r$ is a real number then the set of possible values for the rank of the matrix $\begin{bmatrix} 1 & 1 & r \\ 1 & r & 1 \\ r & 1 & 1 \end{bmatrix}$ is

(a) $1, 2$ (b) $0, 1, 2, 3$ (c) $1, 2, 3$ (d) $0, 1, 2$ (e) $2, 3$

16. Let $\mathcal{B} = \{1 - t, 2 + t^2, t - t^2\}$ be a basis for the vector space $\mathbb{P}_2$ of all polynomials in $t$ of degree at most 2. Find the $\mathcal{B}$-coordinates of $p = 5t - 2t^2$.

(a) $[p]_{\mathcal{B}} = \begin{bmatrix} t-2 \\ 1 \\ 2 \end{bmatrix}$ (b) $[p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}$ (c) $[p]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

(d) $[p]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$ (e) $[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1/2 \\ 2 \end{bmatrix}$

17. Find the maximal interval on which a solution to the initial value problem

$$\begin{cases} t^2 \cdot y'' + \dfrac{t^3}{t^2-3} \cdot y' - \tan(t) \cdot y = \ln|2t+1| \\ y(-1) = \pi/2, \; y'(-1) = 0 \end{cases}$$

is guaranteed to exist.

(a) $-\sqrt{3} < t < \sqrt{3}$ (b) $t < -0.5$ (c) $-\sqrt{3} < t < -0.5$ (d) $\dfrac{-\pi}{2} < t < 0$

(e) $\dfrac{-\pi}{2} < t < -0.5$

18. A matrix with eigenvectors $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$, and corresponding eigenvalues 3 and 2 is

(a) $\begin{bmatrix} 6 & -6 \\ -3 & 2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -3 \\ -1 & -2 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 5 \\ 6 & 0 \end{bmatrix}$

19. The determinant of $\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 4 & 4 & -1 & 6 \\ -2 & 7 & 0 & -5 \end{bmatrix}$ is

(a) 10 (b) $-12$ (c) $-10$ (d) 6 (e) 0

20. The inverse of the matrix $\begin{bmatrix} -3 & -1 & 2 \\ 4 & 1 & -2 \\ 2 & 0 & -1 \end{bmatrix}$ is

(a) $\begin{bmatrix} -3 & 4 & 2 \\ -1 & 1 & 0 \\ 2 & -2 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 2 \\ -2 & -2 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 2 \\ 0 & -2 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -2 \\ 2 & 2 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} -2 & -1 & 0 \\ 4 & 0 & 2 \\ 3 & 1 & -1 \end{bmatrix}$