Math 20580 Name:____________________

Final Exam Instructor:____________________

December 18, 2013 Section:____________________

Calculators are NOT allowed. Do not remove this answer page – you will return the whole exam. You will be allowed 2 hours to do the test. You may leave earlier if you are finished.

There are 25 multiple choice questions worth 6 points each. Record your answers by placing an $\times$ through one letter for each problem on this answer sheet.

Sign the pledge. "On my honor, I have neither given nor received unauthorized aid on this Exam":

1.	a	b	c	d	e	14.	a	b	c	d	e
2.	a	b	c	d	e	15.	a	b	c	d	e
3.	a	b	c	d	e	16.	a	b	c	d	e
4.	a	b	c	d	e	17.	a	b	c	d	e
5.	a	b	c	d	e	18.	a	b	c	d	e
6.	a	b	c	d	e	19.	a	b	c	d	e
7.	a	b	c	d	e	20.	a	b	c	d	e
8.	a	b	c	d	e	21.	a	b	c	d	e
9.	a	b	c	d	e	22.	a	b	c	d	e
10.	a	b	c	d	e	23.	a	b	c	d	e
11.	a	b	c	d	e	24.	a	b	c	d	e
12.	a	b	c	d	e	25.	a	b	c	d	e
13.	a	b	c	d	e						

Part I: Multiple choice questions (7 points each)

1. At time $t = 0$ a tank contains 5 liters of pure water. Then a salt solution containing 0.1 kg of salt per liter is poured into the tank at a constant rate of 2 liters per minute. At the same time well mixed solution is drained from the tank at the same rate of 2 liters per minute.

   How many kilograms of salt will be in the tank after 5 minutes?

   (a) $0.2(1 - e^{-5})$ (b) $0.2(1 - e^{-4})$ (c) $0.5(1 - e^{-2})$
   (d) $0.4(1 - e^{-5})$ (e) $0.5(1 - e^{-4})$

2. Consider three differential equations

   $$\text{(I) } y'' - y' - 2y = 0 \qquad \text{(II) } y'' + 2y' + 2y = 0 \qquad \text{(III) } y'' + 6y' + 9y = 0$$

   For which of these equations does the general solution satisfy $\lim_{t\to\infty} y(t) = 0$ (independent of initial conditions)?

   (a) only for (II) and (III) (b) only (I) and (II) (c) all three
   (d) none (e) only (I)

3. Let $y(t)$ be the solution of the initial value problem

$$y'' - 10y' + 25y = 0 \qquad y(0) = 1 \qquad y'(0) = 4$$

For what value of t does $y(t)$ equal zero?

(a) $t = -1$ (b) $t = 0$ (c) $t = 1$ (d) $t = 5$ (e) $t = -5$

4. Let $y(t)$, for $t > 0$, be the solution of the initial value problem

$$t^2y'' - 3ty' + 3y = 0 \qquad y(1) = 1 \qquad y'(1) = 3$$

Find $y(2)$.
(**Hint:** Note that $y_1(t) = t$ solves the differential equation, but not the initial conditions. Look for a second solution of the form $y_2(t) = v(t)t$.)

(a) 2 (b) -1 (c) 0 (d) 10 (e) 8

5. Consider the differential equation

$$y'' + 4y = 5e^t$$

Note that e^t is a solution. Find the general solution.

(a) $y = A\cos(2t) + B\sin(2t) + e^t$ with A, B arbitrary constants
(b) $y = A\cos(2t) + B\sin(2t) + Ce^t$ with A, B, C arbitrary constants
(c) $y = A\cos(t) + B\sin(t) + e^t$ with A, B arbitrary constants
(d) $y = A\cos(4t) + Be^t$ with A, B arbitrary constants
(e) $y = A\sin(4t) + Be^t$ with A, B arbitrary constants

6. Consider the differential equation

$$y'' - 4y = -4e^{3t}$$

Note that $y_1(t) = e^{2t}$ and $y_2(t) = e^{-2t}$ are solutions of the homogeneous equation $y'' - 4y = 0$. Use the method of variation of parameters to find $u_1(t), u_2(t)$ so that $u_1(t)y_1(t) + u_2(t)y_2(t)$ is a solution of the inhomogeneous equation.

(a) $u_1(t) = -t, \quad u_2(t) = t^5$ (b) $u_1(t) = e^t, \quad u_2(t) = e^{6t}/6$
(c) $u_1(t) = -e^t, \quad u_2(t) = e^{5t}/5$ (d) $u_1(t) = e^{5t}, \quad u_2(t) = e^t$
(e) $u_1(t) = 1, \quad u_2(t) = 1$

7. Let $y(t)$ be the solution of the initial value problem

$$y'' + 3y' - 4y = 0 \qquad y(0) = 1 \qquad y'(0) = \alpha$$

For what values of the parameter α does $\lim_{t\to\infty} y(t) = 0$?

(a) -1 (b) 2 (c) 0 (d) -4 (e) any value of α

8. Consider a logistic population growth model with a threshold given by

$$\frac{dy}{dt} = -0.1(1 - \frac{y}{2})(1 - \frac{y}{7})y$$

where y is measured in millions of bacteria.

Let y_1 be a solution with initial value $y_1(0) = 1$ and y_2 be a solution with initial value $y_2(0) = 8$. Describe the limiting behavior of these solutions as $t \to +\infty$.

(a) $\lim_{t\to\infty} y_1 = 0,\ \lim_{t\to\infty} y_2 = 7$ (b) $\lim_{t\to\infty} y_1 = 0,\ \lim_{t\to\infty} y_2 = \infty$
(c) $\lim_{t\to\infty} y_1 = 2,\ \lim_{t\to\infty} y_2 = 7$ (d) $\lim_{t\to\infty} y_1 = 2,\ \lim_{t\to\infty} y_2 = \infty$
(e) $\lim_{t\to\infty} y_1 = 2,\ \lim_{t\to\infty} y_2 = 2$

9. Find an implicit solution to the initial value problem

$$\cos x \sin y + \sin x \cos y \frac{dy}{dx} = 0, \qquad y(\frac{\pi}{2}) = \frac{\pi}{4}.$$

(a) $\cos y \cos x = \frac{1}{\sqrt{2}}$ (b) $\cos y \sin x = \frac{1}{\sqrt{2}}$ (c) $\sin y \cos x = \frac{1}{\sqrt{2}}$
(d) $\sin y \sin x = \frac{1}{\sqrt{2}}$ (e) $\cos y \cos x = \frac{1}{2}$

10. Which of the following is a general solution to the differential equation

$$1 + (\frac{x}{y} - \sin y)\frac{dy}{dx} = 0?$$

(a) $xy + y \sin y - \sin y = c$ (b) $xy + y \sin y - \cos y = c$
(c) $xy + y \cos y - \sin y = c$ (d) $xy + y \cos y - \cos y = c$
(e) $xy + y \cos y - \sin y = cy$

11. Find the solution to the initial value problem

$$y'' + 4y' + 5y = 0, \qquad y(0) = 0, y'(0) = 1.$$

(a) $y = e^{-t}\sin t$ (b) $y = e^{-t}\cos t$ (c) $y = e^{-2t}\sin t$
(d) $y = e^{-2t}\cos t$ (e) $y = e^{-2t}(\sin t - \cos t)$

12. Let y_1 and y_2 be solutions of the linear homogeneous second order ODE

$$y'' + t^2 y' + t^4 y = 0.$$

Suppose that the Wronskian $W(y_1, y_2)(0) = 1$. Find $W(y_1, y_2)(1)$.

(a) e^{-1} (b) $e^{-1/2}$ (c) $e^{-1/3}$ (d) $e^{-1/4}$ (e) $e^{-1/5}$

13. Find the general solution to the second order ODE

$$y'' + y' = 2.$$

(a) $c_1e^{-t} + c_2te^{-t} + 2t$ (b) $c_1e^t + c_2e^{-t} + 2t$ (c) $c_1e^t + c_2e^{-t} + t$
(d) $c_1 + c_2e^{-t} + t$ (e) $c_1 + c_2e^{-t} + 2t$

14. Find the solution to the initial value problem

$$y' - y = e^t, \qquad y(0) = 2.$$

(a) $y = (t+1)e^t$ (b) $y = (2-t)e^{2t}$ (c) $y = (t+1)e^{2t}$
(d) $y = (t+2)e^{2t}$ (e) $y = (t+2)e^t$

15. Consider the linear system given by $Ax = b$, where

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & h \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

Find all values of h for which this system is consistent.

(a) $h \neq 1$ (b) all h (c) no h (d) $h \neq 0$ (e) $h = 2$

16. Find the least squares solution of $Ax = b$, where

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 0 \\ 3 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$$

(a) $\hat{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ (b) $\hat{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (c) $\hat{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ (d) $\hat{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ (e) $\hat{x} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$

17. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & -2 \\ 2 & 1 & 2 & -5 \\ 3 & 0 & -3 & 0 \\ 4 & 1 & -4 & -1 \end{bmatrix}$$

Which of the following vectors is in the null-space of A?

(a) $\begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}$ (d) $\begin{bmatrix} -1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

18. Let V be the vector space of all polynomials in t with degree at most 2, and $\mathcal{B} = \{1,\ 2t,\ (4+t)^2\}$ a basis for V. Find the $\mathcal{B}$-coordinates of $p = t^2 + 10t + 19$.

(a) $[p]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ (b) $[p]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (c) $[p]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

(d) $[p]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ (e) $[p]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

19. Suppose that A is a 13×10 matrix, and there exists a *non-zero* column vector v such that $Av = 0$.

    What is an impossible value for the rank of A?

    (a) 0 (b) 10 (c) 8 (d) 9 (e) 1

20. Let V be the vector space of all polynomials in t with degree at most 2, and $\mathcal{B} = \{1,\ 2t,\ 2t^2\}$, $\mathcal{C} = \{1,\ 1+t,\ (1+t)^2\}$ be two bases for V. Find the change-of-basis matrix from $\mathcal{C}$ to $\mathcal{B}$.

    (a) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1/2 & 0 \\ 1 & 1 & 1/2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1/2 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -2 & 2 \\ 0 & -2 & -4 \\ 0 & 0 & 2 \end{bmatrix}$

    (d) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $\begin{bmatrix} 1/2 & 1 & 1 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1/2 \end{bmatrix}$

21. The matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

can be diagonalized as $A = PDP^{-1}$. Which of the following matrices could be taken as D?

(a) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ (c) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

(d) $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (e) $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

22. Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ onto the subspace of $\mathbb{R}^3$ spanned by the vectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 \\ -6/13 \\ 4/13 \end{bmatrix}$ (b) $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 2/13 \\ 4/13 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ -4/13 \\ -6/13 \end{bmatrix}$ (e) $\begin{bmatrix} 0 \\ 4/13 \\ -6/13 \end{bmatrix}$

23. Let

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 2 \end{bmatrix}.$$

Find $\det(A^3)$.

(a) 4 (b) 8 (c) 16 (d) 32 (e) 64

24. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}.$$

(a) $\begin{bmatrix} -1/2 & 2 & 1/2 \\ 1 & 2 & 2 \\ -1/2 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -1/2 & 0 & 1 \\ 1 & -1/2 & 0 \\ -1/2 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} -1/2 & 0 & 1/2 \\ 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \end{bmatrix}$

(d) $\begin{bmatrix} 1/2 & 1 & -1/2 \\ 1 & 2 & -1 \\ -1/2 & 2 & 1/2 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

25. Consider the linearly independent vectors in $\mathbb{R}^4$:

$$x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad x_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad x_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Apply the Gram-Schmidt orthogonalization process to x_1, x_2, x_3 to get orthogonal vectors v_1, v_2, v_3. (Do not normalize vectors in the process)

(a) $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

(b) $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

(c) $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

(d) $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

(e) $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}$