1. Find the reduced echelon form of the matrix $\begin{bmatrix} 2 & 1 & 1 & -5 \\ 1 & -2 & 8 & -5 \\ 1 & 1 & -1 & -2 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -3 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
(e) $\begin{bmatrix} 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution. $\begin{bmatrix} 2 & 1 & 1 & -5 \\ 1 & -2 & 8 & -5 \\ 1 & 1 & -1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & -2 \\ 1 & -2 & 8 & -5 \\ 2 & 1 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & -3 & 9 & -3 \\ 0 & -1 & 3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & -2 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$\sim \begin{bmatrix} 1 & 0 & 2 & -3 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ where $\sim$ denotes row equivalence.

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2. Which of the following equations involving 3×3-matrices A, B, C and I_3 (the identity matrix) *could* be *false* for some such matrices A, B, C ?

(a) $(A+B)^2 = A^2 + 2AB + B^2$ (b) $(A+B)C = AC + BC$ (c) $(AB)C = A(BC)$
(d) $A + B = B + A$ (e) $(I_3 + A)(I_3 - A) = I_3 - A^2$

Solution. $(I_3+A)(I_3-A) = I_3(I_3-A)+A(I_3-A) = I_3I_3-I_3A+AI_3-AA = I_3-A+A-A^2 = I_3 - A^2$ so (e) is true.

$(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B) = AA + AB + BA + BB = A^2 + AB + BA + B^2$ could be different from $A^2 + 2AB + B^2$ if $AB \neq BA$, which can happen for some A, B. So (a) could be false.

The other formulae (b), (c), (d) are all standard matrix laws which are true for all 3×3 matrices.

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3. Determine by inspection which of the following sets of vectors is linearly independent.

(a) $\left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}, \right\}$ (c) $\left\{ \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 6 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 7 \end{bmatrix} \right\}$
(d) $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$ (e) $\left\{ \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

Solution. First vector in (a) is non-zero and second is not a scalar multiple of it; so they are linearly independent.

(b) Not linearly independent; $\mathbf{v}_2 = 2\mathbf{v}_1$.
(c) Four vectors in $\mathbb{R}^3$ must be linearly dependent.
(d) Not linearly independent; $\mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2$
(e) Not linearly independent; contains the zero vector.

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4. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by counterclockwise rotation of the plane about the origin by an angle of $\frac{\pi}{4}$ (in radians). Let A be the standard matrix of T. Which of the following matrices is equal to A^2 ?

(a) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Solution. Let $\theta = \frac{\pi}{4}$. One has $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$ and $A^2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ by matrix multiplication. Alternatively, note that A^2 is the matrix of the composite linear transformation given by rotating counterclockwise twice around the origin by angle θ i.e. A^2 is the standard matrix of a rotation by $2\theta = \frac{\pi}{2}$ which is $\begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

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5. Consider the linear system $\begin{bmatrix} 2 & -3 \\ -6 & 9 \\ 4 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ h \\ k \end{bmatrix}$ where h and k are real numbers. Which one of the following statements is true about the solution?

(a) The system is inconsistent if $h \neq -3$. (b) The system is inconsistent if $k \neq 2$.
(c) The system is not consistent for any value of h and k.
(d) The system is consistent for all values of h and k.
(e) For some values of h and k, the system has more than one solution.

Solution. $\begin{bmatrix} 2 & -3 & 1 \\ -6 & 9 & h \\ 4 & -7 & k \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 1 \\ 0 & 0 & h+3 \\ 0 & -1 & k-2 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 & 1 \\ 0 & -1 & k-2 \\ 0 & 0 & h+3 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & 7-3k \\ 0 & -1 & k-2 \\ 0 & 0 & h+3 \end{bmatrix} \sim$ $\begin{bmatrix} 1 & 0 & (7-3k)/2 \\ 0 & 1 & 2-k \\ 0 & 0 & h+3 \end{bmatrix}$. The system is inconsistent precisely when $h+3 \neq 0$ i.e. $h \neq -3$. If $h = -3$, it has the unique solution $x = (7-3k)/2$, $y = 2-k$.

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6. The dimension of the null space of a 7×8 matrix B is 5. How many rows of zeros does the row reduced echelon form of B contain?

(a) 4 (b) 2 (c) 3 (d) 5 (e) 1

Solution. $\operatorname{rank}(B) = 8 - \operatorname{nullity}(B) = 8 - 5 = 3$. The reduced echelon form of B has 3 pivot rows and 7 rows altogether, so there are $7 - 3 = 4$ rows of zeros.

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7. Let $\mathcal{B}$ denote the basis of $\mathbb{R}^3$ given by $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} \right\}$ and let $\mathbf{v}$ denote the vector $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. The coordinate vector $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$ with respect to $\mathcal{B}$ is $[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Which of the following is the value of a?

(a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $-\frac{1}{6}$ (e) 0

Solution. We have to solve the linear system with augmented matrix the first matrix in:
$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 2 & 1 \\ 1 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 3 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$
$\sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/3 \end{bmatrix}$. The coordinate vector is $\begin{bmatrix} 1/3 \\ 0 \\ 1/3 \end{bmatrix}$ so $a = \frac{1}{3}$.

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8. Let A be an $n \times n$ square matrix. Suppose that for some $\mathbf{b}$ in $\mathbb{R}^n$, the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent. Which of the following statements must be true?

(a) The linear system $A\mathbf{x} = \mathbf{c}$ has more than one solution for some $\mathbf{c}$ in $\mathbb{R}^n$.
(b) A has a pivot in every column.
(c) The linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one.
(d) There is an $n \times n$-matrix B with $AB = \mathrm{I}_n$.
(e) The linear system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Solution. Since $A\mathbf{x} = \mathbf{b}$ is inconsistent for some $\mathbf{b}$, the linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\mathbf{x}) = A\mathbf{x}$ is not onto. If T is not onto, the equivalent conditions for matrix invertibility show that A is not invertible, and T is not one-to-one either, so $A\mathbf{x} = \mathbf{c}$ has at least two solutions for some $\mathbf{c}$ in $\mathbb{R}^n$. The conditions for invertibility also show that the other listed conditions (b),(c),(d),(e) are all equivalent to invertibility of A, so cannot hold.

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9. Which of the following is the solution of the matrix equation $\begin{bmatrix} 3 & -1 \\ -17 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$?

(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/2 & -1/2 \\ -17/2 & -3/2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$
(b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3/2 & 1/2 \\ 17/2 & 5/2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$
(c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3/2 & -1/2 \\ -17/2 & -5/2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$
(d) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3/2 & -17/2 \\ -1/2 & -5/2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$
(e) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/2 & -17/2 \\ -1/2 & -3/2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$

Solution. The equation is $A\mathbf{v} = \mathbf{b}$ where $A = \begin{bmatrix} 3 & -1 \\ -17 & 5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} h \\ k \end{bmatrix}$. Note $\det A = 3 \cdot 5 - (-1) \cdot (-17) = -2$ so A is invertible and $A^{-1} = \frac{1}{-2} \begin{bmatrix} 5 & 1 \\ 17 & 3 \end{bmatrix}$. The solution is $\mathbf{v} = A^{-1}\mathbf{b}$ i.e. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/2 & -1/2 \\ -17/2 & -3/2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$.

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10. Compute the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ -3 & -2 & -6 \\ -1 & -1 & -2 \end{bmatrix}$.

Solution. Row-reduce $\begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ -3 & -2 & -6 & 0 & 1 & 0 \\ -1 & -1 & -2 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \sim$ $\begin{bmatrix} 1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 1 & 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$. The inverse is $\begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & -3 \\ 1 & 0 & 1 \end{bmatrix}$.

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11. Express the solution set of

$$\begin{array}{rcrcrcrcr} 2x_1 & - & 4x_2 & + & 5x_3 & + & x_4 & = & -3 \\ x_1 & - & 2x_2 & + & 2x_3 & + & x_4 & = & -1 \\ x_1 & - & 2x_2 & + & 3x_3 & & & = & -2 \end{array}$$

in *parametric vector form.*

Solution. Row reduce: $\begin{bmatrix} 2 & -4 & 5 & 1 & -3 \\ 1 & -2 & 2 & 1 & -1 \\ 1 & -2 & 3 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 1 & -1 \\ 2 & -4 & 5 & 1 & -3 \\ 1 & -2 & 3 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 2 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}$ $\sim \begin{bmatrix} 1 & -2 & 0 & 3 & 1 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. The equation is equivalent to

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & & & + & 3x_4 & = & 1 \\ & & & & x_3 & - & x_4 & = & -1 \end{array}$$

where the free variables x_2 and x_4 can take arbitrary values (the last row of the matrix gives the equation $0 = 0$, which we omit because it is always true).

The bound variables are x_1, x_3 (corresponding to pivot columns) and the free variables are x_2, x_4. Rewriting with free variables on the right,

$$\begin{array}{rcrcrcr} x_1 & = & 1 & + & 2x_2 & - & 3x_4 \\ x_2 & = & & & x_2 & & \\ x_3 & = & -1 & & & + & x_4 \\ x_4 & = & & & & & x_4 \end{array}$$

(we include the equation $x_i = x_i$, for $i = 2$ or 4, to indicate that the free variable x_i can take arbitrary values). In parametric form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

or writing $x_2 = r$, $x_4 = s$,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

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12. The row-reduced echelon form of the 3×6 matrix $A = \begin{bmatrix} 0 & 2 & 4 & 1 & 5 & 6 \\ 0 & 1 & 2 & -1 & 7 & -5 \\ 0 & -1 & -2 & -2 & 2 & 0 \end{bmatrix}$ is given by $B = \begin{bmatrix} 0 & 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. (You may assume this; you do not have to check it.)

(a) Determine a basis for the null space null(A).

(b) Determine a basis for the column space col(A).

Solution. (a) The pivot columns of A and B are 2, 4, 6, so x_1, x_3 and x_5 are free variables. Writing the homogeneous equations from B with free variables on the right gives

$$\begin{aligned} x_1 &= x_1 \\ x_2 &= -2x_3 - 4x_5 \\ x_3 &= x_3 \\ x_4 &= 3x_5 \\ x_5 &= x_5 \\ x_6 &= 0 \end{aligned}$$

We include the equation $x_i = x_i$, for $i = 1$, 3 or 5, to indicate that the free variable x_i can take arbitrary values. The system has 3 basic solutions given by setting one free variable equal to 1 and the others equal to 0. Setting $x_1 = 1$ and $x_3 = x_5 = 0$ gives the solution $\mathbf{v_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T$. Setting $x_3 = 1$ and $x_1 = x_5 = 0$ gives the solution $\mathbf{v_2} = \begin{bmatrix} 0 & -2 & 1 & 0 & 0 & 0 \end{bmatrix}^T$. Setting $x_5 = 1$ and $x_1 = x_3 = 0$ gives the solution $\mathbf{v_3} = \begin{bmatrix} 0 & -4 & 0 & 3 & 1 & 0 \end{bmatrix}^T$. Then $\{ \mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3} \}$ is a basis for null(A).

(b) Row operations don't change the solution space of the homogeneous equation or the linear dependences of columns of a matrix. The pivot columns (2rd, 4th, 6th) of B form a basis for col(B) so the pivot columns (2rd, 4th, 6th) of A form a basis for col(A). A basis of col(A) is given by $\{ \mathbf{w_1}, \mathbf{w_2}, \mathbf{w_3} \}$ where $\mathbf{w_1} = \begin{bmatrix} 2 & 1 & -1 \end{bmatrix}^T$, $\mathbf{w_2} = \begin{bmatrix} 1 & -1 & -2 \end{bmatrix}^T$ and $\mathbf{w_3} = \begin{bmatrix} 6 & -5 & 0 \end{bmatrix}^T$.