## Multiple Choice

**1.** (6 pts.) Find the reduced echelon form of the matrix $\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & 0 \\ -3 & -6 & 1 & 0 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$ (e) $\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

**2.** (6 pts.) Determine by inspection which of the following sets is linearly independent.

(a) $\left\{ \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ -2 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} 4 \\ -6 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ -9 \\ 3 \\ -6 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \\ 0 \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \right\}$

(e) all four sets are linearly dependent

**3.** (6 pts.) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ and $S : \mathbb{R}^3 \to \mathbb{R}^2$ be linear transformations with

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \text{ and}$$

$$S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad S\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad S\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Which matrix below is the standard matrix of $ST$?

(a) $\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 0 & -2 & 0 \\ 2 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ (e) $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$

**4.** (6 pts.) The determinant of $\begin{bmatrix} 0 & 2 & -3 \\ -2 & 6 & -12 \\ 1 & -2 & 3 \end{bmatrix}$ is

(a) $-6$ (b) $-12$ (c) $0$ (d) $6$ (e) $12$

**5.** (6 pts.) Let $A$ be a $7 \times 8$ matrix of rank 3. Which of the following is equal to the dimension of the null space of $A$?

(a) 5 (b) 0 (c) 3 (d) 4 (e) 7

**6.** (6 pts.) Let $\mathcal{B}$ be the basis of $\mathbb{R}^3$ given by the vectors $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} \right\}$ and let $x$ be the vector $x = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}$. Which of the following is the coordinate vector $[x]_{\mathcal{B}}$ of $x$ with respect to $\mathcal{B}$?

(a) $\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} 14 \\ 0 \\ 6 \end{bmatrix}$ (c) $\begin{bmatrix} 2 \\ 3 \\ -2 \end{bmatrix}$

(d) $\begin{bmatrix} 4 \\ -1 \\ 3 \end{bmatrix}$ (e) $\begin{bmatrix} -2 \\ 0 \\ 5 \end{bmatrix}$

**7.** (6 pts.) Suppose an $n \times n$ square matrix $A$ is such that the homogeneous linear system $Ax = 0$ has a non-trivial solution. Which of the following statements must be true?

(a) The linear system $Ax = b$ is inconsistent for some $b$ in $\mathbb{R}^n$

(b) $A$ has a pivot in every column.

(c) The linear map $T\colon \mathbb{R}^n \to \mathbb{R}^n$ given by $T(x) = Ax$ is onto.

(d) There is an $n \times n$-matrix $B$ with $AB = \mathrm{I}_n$.

(e) The linear system $A^T x = 0$ has only the trivial solution.

**8.** (6 pts.) Which of the following is the solution for $\begin{bmatrix} x \\ y \end{bmatrix}$ of the matrix equation

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}?$$

(a) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$

(b) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5/2 & -3/2 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$

(c) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & 3/2 \\ 2 & -5/2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$

(d) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 & -3/2 \\ -2 & -5/2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$

(e) $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$

**9.** (6 pts.) Let $A = \begin{bmatrix} 2 & 3 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 2 & -1 & -2 \end{bmatrix}$. What is the rank of $A$?

(a) 2 (b) 0 (c) 1 (d) 3 (e) 4

## Partial Credit

You must show your work on the partial credit problems to receive credit!

**10.** (14 pts.) Express the solution set of

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & -6 \\ x_1 & & & + & x_3 & + & x_4 & = & -3 \\ x_1 & - & x_2 & + & 3x_3 & & & = & 0 \end{array}$$

in *Parametric Vector Form.*

**11.** (14 pts.) The row-reduced echelon form of the $3 \times 5$ matrix $A = \begin{bmatrix} 2 & -4 & 1 & 1 & 5 \\ 3 & -6 & -2 & 5 & -7 \\ 5 & -10 & 3 & 2 & 4 \end{bmatrix}$ is given by $B = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. (You may assume this; you do not have to check it.)

(a) Determine a basis for the null space null($A$).
(b) Determine a basis for the column space col($A$).
(c) Determine a basis for the row space row($A$).

**12.** (14 pts.) Compute the inverse of the matrix $A = \begin{bmatrix} 4 & 2 & 3 \\ 2 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

## Exam 1D solutions

**Multiple choice.** Exam 1D has all multiple choice answers (a).

(1) $\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & 0 \\ -3 & -6 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(2) (a), clearly one vector is not a multiple of the other so LI.
(b): too many vectors - dependent.
(c): 3rd vector is a multiple of first- dependent.
(d): zero vector - dependent.
(e): false since LI in (a).

(3) The standard matrix of $T$ is $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$. The standard matrix of $S$ is $B = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$. Hence the standard matrix of $ST$ is the matrix product $BA = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$

(4) $\begin{vmatrix} 0 & 2 & -3 \\ -2 & 6 & -12 \\ 1 & -2 & 3 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 3 \\ -2 & 6 & -12 \\ 0 & 2 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 3 \\ 0 & 2 & -6 \\ 0 & 2 & -3 \end{vmatrix} = -\begin{vmatrix} 1 & -2 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{vmatrix}$
Since this last matrix is upper triangular, the determinant is $-1 \cdot 2 \cdot 3 = -6$.

(5) For a $p \times q$-matrix, $\text{rank}(A) + \dim(\text{null}(A)) = q$. Hence $\dim(\text{null}(A)) = q - \text{rank}(A) = 8 - 3 = 5$.

(6) We have to solve the linear system whose augmented matrix is first matrix following: $\begin{bmatrix} 1 & 1 & 2 & 4 \\ -2 & 0 & 2 & 2 \\ 1 & 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 2 & 6 & 10 \\ 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$
$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. The coordinate vector is $\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}$.

(7) $Ax = b$ is inconsistent for some $b$, directly from the invertible matrix theorem.

(8) Let $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$. Note $A$ has determinant $2 \cdot 5 - 3 \cdot 4 = -2$ and so is invertible. The solution is $\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1} \begin{bmatrix} h \\ k \end{bmatrix}$. So $\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-2} \begin{bmatrix} 5 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} -5/2 & 3/2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix}$.

(9) Row reduce: $\begin{bmatrix}2&3&0&0\\1&1&1&2\\1&2&-1&-2\end{bmatrix}$ $\begin{bmatrix}1&1&1&2\\2&3&0&0\\1&2&-1&-2\end{bmatrix}$ $\begin{bmatrix}1&1&1&2\\0&1&-2&-4\\0&1&-2&4\end{bmatrix}$ $\begin{bmatrix}1&0&3&6\\0&1&-2&-4\\0&0&0&0\end{bmatrix}$.

**Partial credit**

(10) $\begin{bmatrix}1&1&-1&2&-6\\1&0&1&1&-3\\1&-1&3&0&0\end{bmatrix}$ $\begin{bmatrix}1&1&-1&2&-6\\0&-1&2&-1&3\\1&-2&4&-2&6\end{bmatrix}$ $\begin{bmatrix}1&1&-1&2&-6\\0&1&-2&1&-3\\0&0&0&0&0\end{bmatrix}$ $\begin{bmatrix}1&0&1&-1&-3\\0&1&-2&1&-3\\0&0&0&0&0\end{bmatrix}$. The equation is equivalent to

$$\begin{array}{ccccccccc}x_1 & & & + & x_3 & - & x_4 & = & -3\\ & & x_2 & - & 2x_3 & + & x_4 & = & -3\end{array}$$

The bound variables are $x_1$, $x_2$, and free variables are $x_3$, $x_4$. Rewriting with free variables on the right,

$$\begin{array}{ccccccc}x_1 & & = & -3 & -x_3 & - & x_4\\ & x_2 & = & -3 & +2x_3 & - & x_4\end{array}$$

or in vector parametric form

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-3\\-3\\0\\0\end{bmatrix} + x_3\begin{bmatrix}-1\\2\\1\\0\end{bmatrix} + x_4\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix} =$$

or writing $x_3 = a$, $x_4 = b$,

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-3\\-3\\0\\0\end{bmatrix} + a\begin{bmatrix}-1\\2\\1\\0\end{bmatrix} + b\begin{bmatrix}-1\\-1\\0\\1\end{bmatrix}$$

(11) (a) The pivot columns of $A$ and $B$ are 1, 3, 5, so $x_2$ and $x_4$ are free variables. Writing the homogeneous equations from $B$ with free variables on the right gives $\begin{array}{ccccc}x_1 & = & 2x_2- & x_4\\ x_3 & = & & x_4\\ x_5 & = & 0 & \end{array}$. The system has 2 basic solutions given by setting one free variable equal to 1 and the others equal to 2. Setting $x_2 = 1$ and $x_4 = 0$ gives the solution $v_1 = \begin{bmatrix}2&1&0&0&0\end{bmatrix}^T$. Setting $x_2 = 0$ and

$x_4 = 1$ gives the solution $v_2 = \begin{bmatrix}-1 & 0 & 1 & 1 & 0\end{bmatrix}^T$ (we write these using transpose $T$ to save space). Then $\{ v_1, v_2 \}$ is a basis for null$(A)$.

(b) Row operations don't change the solution space of the homogeneous equation or the linear dependences of columns of a matrix. The pivot columns (1st, 2rd, 5th) of $B$ form a basis for col$(B)$ so the pivot columns (1st, 2rd, 5th) of $A$ form a basis for col$(A)$. A basis of col$(A)$ is given by $\{ w_1, w_2, w_3 \}$ where $w_1 = \begin{bmatrix}2\\3\\5\end{bmatrix}$ and $w_2 = \begin{bmatrix}1\\-2\\3\end{bmatrix}$, $w_3 = \begin{bmatrix}5\\-7\\4\end{bmatrix}$.

(c) Row span of a matrix is unchanged by ERO's, so row$(A) =$ row$(B)$. Since $B$ is in echelon form, its non-zero rows form a basis of row$(B)$ and hence of row$(A)$. So a basis of row$(A)$ is given by $\{ u_1, u_2, u_3 \}$ where $u_1 = \begin{bmatrix}1 & -2 & 0 & 1 & 0\end{bmatrix}$, $u_2 = \begin{bmatrix}0 & 0 & 1 & -1 & 0\end{bmatrix}$ and $u_1 = \begin{bmatrix}0 & 0 & 0 & 0 & 1\end{bmatrix}$.

(12) Row-reduce: $\begin{bmatrix}4 & 2 & 3 & 1 & 0 & 0\\2 & 2 & 2 & 0 & 1 & 0\\1 & 0 & 1 & 0 & 0 & 1\end{bmatrix}$ $\begin{bmatrix}1 & 0 & 1 & 0 & 0 & 1\\2 & 2 & 2 & 0 & 1 & 0\\4 & 2 & 3 & 1 & 0 & 0\end{bmatrix}$
$\begin{bmatrix}1 & 0 & 1 & 0 & 0 & 1\\0 & 2 & 0 & 0 & 1 & -2\\0 & 2 & -1 & 1 & 0 & -4\end{bmatrix}$ $\begin{bmatrix}1 & 0 & 1 & 0 & 0 & 1\\0 & 2 & 0 & 0 & 1 & -2\\0 & 0 & -1 & 1 & -1 & -2\end{bmatrix}$ $\begin{bmatrix}1 & 0 & 0 & 1 & -1 & -1\\0 & 2 & 0 & 0 & 1 & -2\\0 & 0 & -1 & 1 & -1 & -2\end{bmatrix}$
$\begin{bmatrix}1 & 0 & 0 & 1 & -1 & -1\\0 & 1 & 0 & 0 & 1/2 & -1\\0 & 0 & 1 & -1 & 1 & 2\end{bmatrix}$ so $\begin{bmatrix}1 & -1 & -1\\0 & 1/2 & -1\\-1 & 1 & 2\end{bmatrix}$ is the inverse.