Useful Product Formulas

Formula	Example
$(a+b)^2 = a^2+2ab+b^2$	$(x+3y)^2 = x^2+6xy+9y^2$
$(a-b)^2 = a^2-2ab+b^2$	$(2x-y)^2 = 4x^2-4xy+y^2$
$(a+b)(a-b) = a^2-b^2$	$(x+2)(x-2) = x^2-4$

Factoring

Def. *Factoring* is the process of expressing an algebraic expression as the product of other algebraic expressions.

Ex: 1) $2x^{3/2} - 3x^{1/2} = x^{1/2}(2x-3)$

2) $4x(x+1)^{1/2} - x^2(x+1)^{-1/2}$

$= x(x+1)^{-1/2}\left(4(x+1)-x\right)$

$= x(x+1)^{-1/2}(3x+4)$

3) $2ax + 2ay + bx + by$

$= 2a(x+y) + b(x+y)$

$= (2a+b)(x+y)$

Product Formulas Used in Factoring

Formula	Example
$x^2 - y^2 = (x+y)(x-y)$	$x^2 - 81 = (x+9)(x-9)$
$x^2 + 2xy + y^2 = (x+y)^2$	$x^2 + 8x + 16 = (x+4)^2$
$x^2 - 2xy + y^2 = (x-y)^2$	$4x^2 - 4xy + y^2 = (2x-y)^2$
$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$	$z^3 + 27 = z^3 + 3^3 = (z+3)(z^2 - 3z + 9)$
$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$	$8x^3 - y^6 = (2x)^3 - (y^2)^3 = (2x - y^2)(4x^2 + 2xy^2 + y^4)$

The factors of 2nd degree polynomial w/ integer coefficient

$$px^2 + qx + r$$

are

$$(ax+b)(cx+d)$$

such that

$$ac = p, \quad ad + bc = q, \quad bd = r$$

Ex: 1) $x^2 - 2x - 3$

$= (x \quad)(x \quad)$ product $= -3$, sum $= -2$ $\Rightarrow -3, 1$

$= (x-3)(x+1)$

2) $-3t^2 + 192t + 195$

$= -3(t^2 - 64t - 65)$

$= -3(t \quad)(t \quad)$ $p = -65$, $s = -64$ $\Rightarrow -65, 1$

$= -3(t+1)(t-65)$

Roots of Polynomial Equations

Def. Polynomial of degree n (n nonnegative integer) and variable x:

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

for $a_0, \ldots, a_n$ real numbers

$a_n \neq 0$.

Def. Roots of a polynomial are values of x that satisfy the given eqn.

Ex: Find the roots of $x^3 - 3x^2 + 2x = 0$.

Factor: $x^3 - 3x^2 + 2x = x(x^2 - 3x + 2)$ $\quad P = 2,\ S = -3 \Rightarrow -1, -2$

$= x(x-1)(x-2)$

$\Rightarrow$ roots are $x = 0, 1, 2$

Quadratic Formula

The solutions of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Ex: 1) $x^2 - 3x + 2 = 0$

roots: $x = \dfrac{3 \pm \sqrt{9 - 4(1)(2)}}{2(1)}$

$= \dfrac{3 \pm \sqrt{1}}{2} = \dfrac{3 \pm 1}{2} = 2, 1$

2) $x^2 = -3x + 8 \Rightarrow x^2 + 3x - 8 = 0$

roots: $x = \dfrac{-3 \pm \sqrt{3^2 - 4(1)(-8)}}{2(1)}$

$= \dfrac{-3 \pm \sqrt{9 + 32}}{2} = \dfrac{-3 \pm \sqrt{41}}{2}$

$\approx 1.7, -4.7$

Precalc Review (cont.)

Rational Expressions

Def. Quotients of polynomials are called rational expressions.

Ex: $\dfrac{3x^2y^3 - 2xy}{4x}$ $\leftarrow p_1(x)$ / $\leftarrow p_2(x)$

Note: Can use same properties of fractions (e.g. addition, multiplication, division, simplification, ...)

If P, Q, R and S are polynomials, then

- Multiplication: $\dfrac{P}{Q} \cdot \dfrac{R}{S} = \dfrac{PR}{QS}$
- Division: $\dfrac{P}{Q} \div \dfrac{R}{S} = \dfrac{P}{Q} \cdot \dfrac{S}{R} = \dfrac{PS}{QR}$

  $(Q, R, S \neq 0)$

Ex: Perform indicated operation and simplify expression

$$\frac{2x-8}{x+2} \cdot \frac{x^2+4x+4}{x^2-16} = \frac{2(x-4)}{x+2} \cdot \frac{(x+2)^2}{(x+4)(x-4)}$$

$$= \frac{2\cancel{(x-4)}(x+2)^{\cancel{2}}}{\cancel{(x+2)}(x+4)\cancel{(x-4)}}$$

$$= \frac{2(x+2)}{x+4}$$

⚠ $\dfrac{5+x^2}{5} \neq x^2$

$$= \frac{5}{5} + \frac{x^2}{5} = 1 + \frac{x^2}{5}$$

- Addition: $\frac{P}{R} + \frac{Q}{R} = \frac{P \pm Q}{R}$ $(R \neq 0)$
  Subtraction $-$

Ex: Simplify $\frac{1}{x+h} - \frac{1}{x}$

$\frac{1}{x+h} - \frac{1}{x}$ common denom $\frac{1}{x+h} \cdot \frac{x}{x} - \frac{1}{x} \cdot \frac{(x+h)}{(x+h)}$

$= \frac{x}{x(x+h)} - \frac{(x+h)}{x(x+h)}$

$= \frac{-h}{x(x+h)}$

⚠ $\frac{7}{x^2+x} \neq \frac{7}{x^2} + \frac{7}{x}$

(More) Ex: Perform indicated operation and simplify expression

1) $\dfrac{1 + \frac{1}{x+1}}{x - 4x^{-1}} = \dfrac{\frac{x+1}{x+1} + \frac{1}{x+1}}{x - \frac{4}{x}}$ $\} \rightarrow \frac{x^2}{x} - \frac{4}{x}$

$= \dfrac{\frac{x+2}{x+1}}{\frac{x^2-4}{x}}$

$= \frac{x+2}{x+1} \div \frac{x^2-4}{x}$ $\} \rightarrow (x+2)(x-2)$

$= \frac{x+2}{x+1} \cdot \frac{x}{(x+2)(x-2)}$

$= \frac{\cancel{(x+2)}x}{(x+1)\cancel{(x+2)}(x-2)}$

$= \frac{x}{(x+1)(x-2)}$

2) $\dfrac{12x^2}{\sqrt{2x^2+3}} + 6\sqrt{2x^2+3}$

$= \dfrac{12x^2}{\sqrt{2x^2+3}} + \dfrac{6(2x^2+3)}{\sqrt{2x^2+3}}$

$= \dfrac{12x^2+6(2x^2+3)}{\sqrt{2x^2+3}}$ } $\rightarrow$ $12x^2 + 12x^2 + 18$

$= \dfrac{24x^2+18}{\sqrt{2x^2+3}}$ } $\rightarrow$ $6(4x^2+3)$

=

Rationalizing Algebraic Fractions

Trick : $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2$

$= a-b$

Ex: Rationalize denominator : $\dfrac{1}{1+\sqrt{x^2+2}}$

$\dfrac{1}{1+\sqrt{x^2+2}} \cdot \dfrac{(1-\sqrt{x^2+2})}{(1-\sqrt{x^2+2})}$

$= \dfrac{1-\sqrt{x^2+2}}{1-(x^2+2)}$

$= \dfrac{1-\sqrt{x^2+2}}{-x^2-1}$

Inequalities

Properties: If A, B, C are real numbers or polynomials with real coeff $^{(*)}$, then

1) $A \underline{<} B$ and $B \underline{<} C \Rightarrow A \underline{<} C$
2) $A \underline{<} B \Rightarrow A + C \underline{<} B + C$
3) $A \leq B$ and C real number $> 0 \Rightarrow AC \underline{<} BC$
4) $A \underline{<} B$ and C " " $< 0 \Rightarrow AC \geq BC$

Ex: 1) Find all values of x that satisfy

$-1 \leq 2x - 5 < 7$

$\Rightarrow -1 + 5 \leq 2x - 5 + 5 < 7 + 5$

$\Rightarrow 4 \leq 2x < 12$

$\Rightarrow 4/2 \leq 2x/2 < 12/2$

$\Rightarrow 2 \leq x < 6$

solution: $[2, 6)$

2) Solve inequality $\frac{x+1}{x-1} \geq 0$

$\frac{x+1}{x-1} > 0$ if $x+1$ and $x-1$ have same sign

$(x+1)$	$-$	$-$	0	$+$	$+$	$+$	......		
$(x-1)$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$	...
	-4	-3	-2	-1	0	1	2	3	4

$\Rightarrow$ same sign when x is in $(-\infty, -1)$ and $(1, \infty)$

$\frac{x+1}{x-1} = 0$ if $x + 1 = 0 \Rightarrow x = -1$

$\Rightarrow \frac{x+1}{x-1} \geq 0$ when x is in $(-\infty, -1]$ or $(1, \infty)$