Ex: A manufacturer has monthly fixed cost of \$40,000 and a production cost of \$8 for each unit produced. The product sells for \$12 / unit.

a) What is the cost function (per month)?

$C(x) = 8x + 40{,}000$

\# units produced

production cost per unit

fixed monthly cost

b) What is the revenue function?

$R(x) = 12x$

amount earned per unit

c) What is the profit function?

$$P(x) = R(x) - C(x)$$
$$= 12x - (8x + 40{,}000)$$
$$= 4x - 40{,}000$$

d) Compute the profit / loss corresponding to production levels of 8000 and 12,000 units.

- $P(8000) = 4(8000) - 40{,}000$
  $= 32{,}000 - 40{,}000$
  $= -6{,}000$
  $\Rightarrow$ loss of \$6000
- $P(12{,}000) = 4(12{,}000) - 40{,}000$
  $= 48{,}000 - 40{,}000$
  $= 8{,}000$
  $\Rightarrow$ profit of \$8000

Rational and Power Functions

Def. Rational function := quotient of 2 polynomials

Ex: $F(x) = \dfrac{x^3 + 2x^2 - x + 1}{x+2}$ $\begin{array}{l} \leftarrow f(x) \\ \leftarrow g(x) \end{array}$

$\Rightarrow F(x) = \dfrac{f(x)}{g(x)}$

Power function := functions of the form $f(x) = x^r$, where r is a real #.

Ex: 1) $f(x) = \sqrt{x} = x^{1/2}$

2) $g(x) = \dfrac{1}{x^2} = x^{-2}$

2.4 Limits

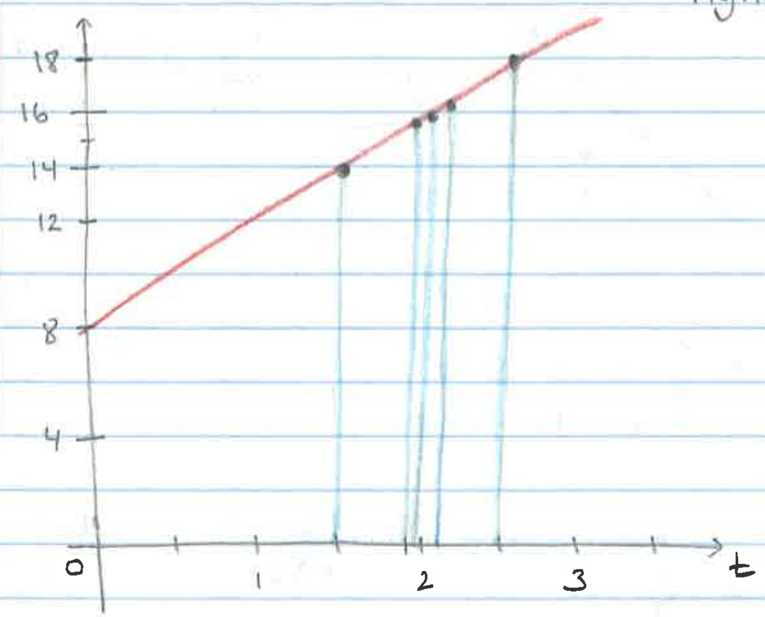
Intuitive Def: Consider function $g(t) = \frac{4(t^2-4)}{t-2}$.

What is the value of g as t approaches 2?

t	1.5	1.9	1.99	2.001	2.01	2.1	2.5
$g(t)$	14	15.6	15.96	16.004	16.04	16.4	18

from the left → ← from the right

⇒ when t approaches 2 from the left, $g(t)$ approaches 16
" " " " " " right, " " "



Def. Function f has limit L as x approaches a:

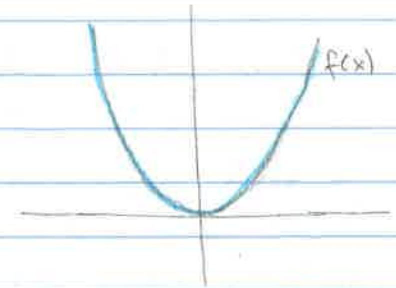
$$\lim_{x\to a} f(x) = L$$

if we can get $f(x)$ as close to L as we want by choosing x sufficiently close (<u>but not equal</u>) to a.

Ex: Suppose we defined $g(t) = \begin{cases} 0, & \text{if } t=2 \\ \frac{4(t^2-4)}{t-2}, & \text{otherwise.} \end{cases}$

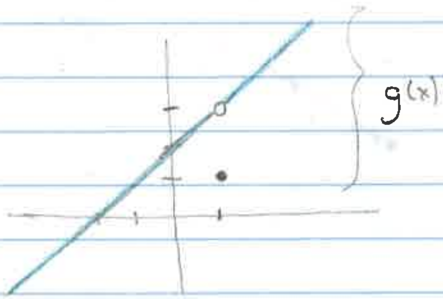
Then we still have $\lim_{t\to 2} g(t) = 16$.

Ex: 1) $f(x) = x^2$



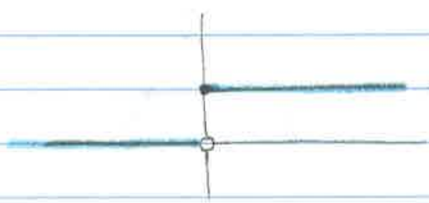
$\lim_{x \to 0} f(x) = 0$

2) $g(x) = \begin{cases} x+2 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$



$\lim_{x \to 1} g(x) = 3$

3) $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$

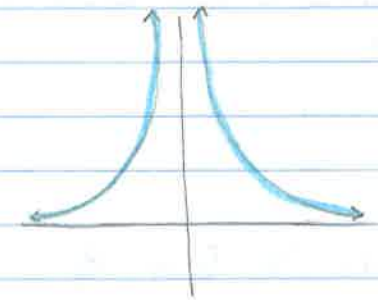


$\lim_{x \to 0} f(x)$ does not exist

Rule: If f(x) does not approach the same value from the right and from the left at given $x = a$, then

$\lim_{x \to a} f(x)$ does not exist

4) $g(x) = \frac{1}{x^2}$



$\lim_{x \to 0} g(x) = \infty$ (my claim)
$= \text{DNE}$ (book's claim)

(I think book wants finite value for a limit)

Properties of Limits

Suppose $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$. Then:

1) $\lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = L \pm M$

2) $\lim_{x \to a} [f(x)\, g(x)] = [\lim_{x \to a} f(x)][\lim_{x \to a} g(x)] = LM$

3) $\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{L}{M}$ (provided $M \neq 0$)

4) $\lim_{x \to a} c\, f(x) = c \lim_{x \to a} f(x) = cL$ (c = a real number)

5) $\lim_{x \to a} [f(x)]^r = \left[\lim_{x \to a} f(x)\right]^r = L^r$ (r = a positive constant)

Ex: 1) $\lim_{x \to 2} \frac{x^2 + 4x + 7}{x^2 + 1} = \frac{\lim_{x \to 2} x^2 + 4x + 7}{\lim_{x \to 2} x^2 + 1} = \frac{19}{5}$

2) $\lim_{x \to 3} 2x^3 \sqrt{x^2 + 7} = \left[\lim_{x \to 3} 2x^3\right]\left[\lim_{x \to 3} (x^2+7)^{1/2}\right]$

$= \left[2 \lim_{x \to 3} x^3\right]\left[\lim_{x \to 3} (x^2+7)\right]^{1/2}$

$= \underbrace{[2 \cdot 3^3]}_{2 \cdot 27 = 54}\underbrace{[3^2 + 7]^{1/2}}_{16^{1/2} = 4}$

$= 54 \cdot 4 = 216$

Intermediate Forms

Recall: Division property of limits is only valid when the denominator $\neq 0$

Consider: $\lim\limits_{x \to 2} \dfrac{4(x^2-4)}{x-2}$

We obtain intermediate form $\frac{0}{0}$.

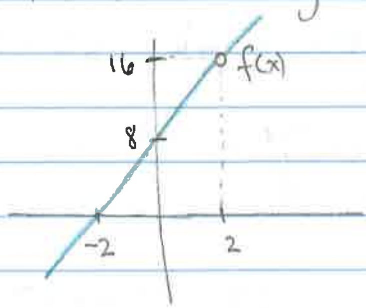
Strategy: "cancel out the zero" if possible:

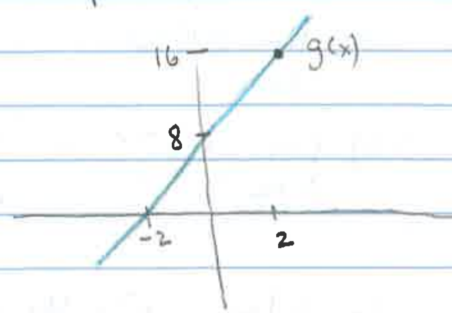
$$\frac{4(x^2-4)}{x-2} = \frac{4(x+2)(x-2)}{\cancel{x-2}} = 4(x+2)$$

$$\Rightarrow \lim_{x \to 2} \frac{4(x^2-4)}{x-2} = \lim_{x \to 2} 4(x+2) = 16$$

Note: Graphs of $f(x) = \dfrac{4(x^2-4)}{x-2}$ and $g(x) = 4(x+2)$

are the same everywhere except $x=2$:





Ex: $\lim\limits_{h \to 0} \dfrac{\sqrt{1+h}-1}{h} = \lim\limits_{h \to 0} \dfrac{\sqrt{1+h}-1}{h} \cdot \dfrac{\sqrt{1+h}+1}{\sqrt{1+h}+1}$

$$= \lim_{h \to 0} \frac{(\cancel{1}+h)-\cancel{1}}{h(\sqrt{1+h}+1)}$$

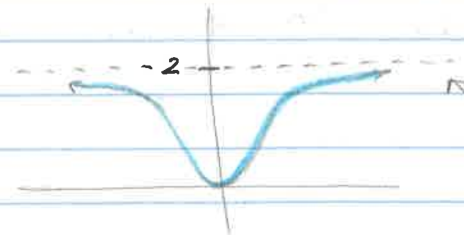
$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{1+h}+1)}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{1+h}+1}$$

$$= \frac{1}{2}$$

Limits at Infinity

Ex: $\lim\limits_{x\to\infty} \frac{2x^2}{1+x^2} = 2$



horizontal asymptote at $y=2$

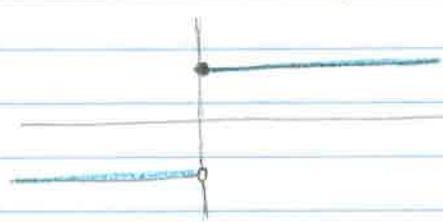
The function f has limit L as x increases w/o bound $(x \to \infty)$:

$$\lim_{x\to\infty} f(x) = L$$

if $f(x)$ can be made arbitrarily close to L by taking x large enough

* Similarly for $\lim\limits_{x\to-\infty} f(x) = L$, where x decreases w/o bound and we need x small enough

Ex: $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$



$\lim\limits_{x\to+\infty} f(x) = 1$

$\lim\limits_{x\to-\infty} f(x) = -1$

<u>Theorem</u>: For all $n > 0$

$\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{x^n} = 0$

(provided $\frac{1}{x^n}$ is defined)

Ex: $\lim_{x\to\infty} \frac{x^2-x+3}{2x^3+1} = \lim_{x\to\infty} \frac{x^2-x+3}{2x^3+1} \cdot \frac{\frac{1}{x^3}}{\frac{1}{x^3}}$

$= \lim_{x\to\infty} \frac{\frac{1}{x} - \frac{1}{x^2} + \frac{3}{x^3}}{2 + \frac{1}{x^3}}$

$= \frac{0}{2} = 0$