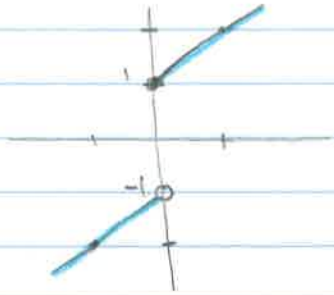
2.5 One-Sided Limits and Continuity

Consider $f(x) = \begin{cases} x-1 & \text{if } x < 0 \\ x+1 & \text{if } x \geq 0 \end{cases}$



$\lim_{x \to 0^+} f(x) = 1$

$\lim_{x \to 0^-} f(x) = -1$

Def. Function f has <u>right-hand limit</u> L as x approaches a from the right

$$\lim_{x \to a^+} f(x) = L$$

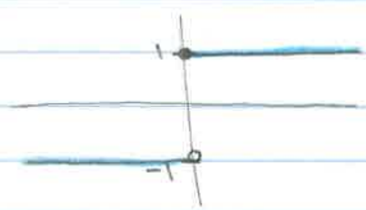
if the values of f can be made as close as we want to L by taking x sufficiently close (but not equal to) a and to the right of a.

Similarly for <u>left-hand limit</u>.

<u>Theorem</u>: Let f be defined for all values of x close to $x=a$ (possibly excluding $x=a$ itself). Then

$$\lim_{x \to a} f(x) = L \text{ iff } \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

Ex. Recall $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$



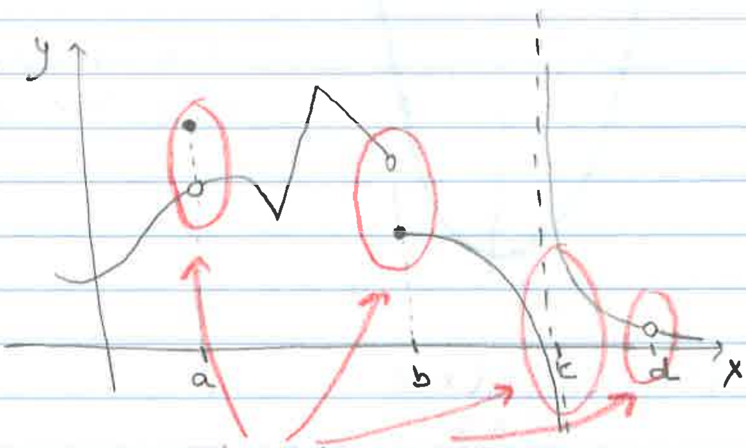
Note $\lim_{x \to 0^-} f(x) = -1$ and $\lim_{x \to 0^+} f(x) = 1$

$\overset{\text{theorem}}{\Longrightarrow} \lim_{x \to 0} f(x)$ DNE

Continuous Functions

Informally: A function is continuous at a point if the graph of the function at that point has no holes, gaps, jumps, or breaks.

Ex:



Def. A function f is continuous at $x=a$ if the following conditions are satisfied:

1) $f(a)$ is defined
2) $\lim_{x \to a} f(x)$ exists
3) $\lim_{x \to a} f(x) = f(a)$

Def. If f is not continuous at $x=a$, then it is discontinuous at $x=a$.

Def. f is continuous on an interval if f is continuous at every number on that interval.

Ex: $(0, a)$? ✓

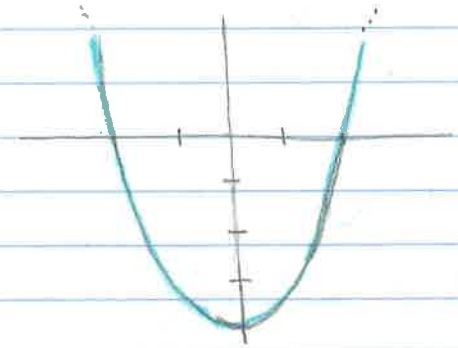
$[0, a]$? X

(a, c)? X

(c, d)? ✓

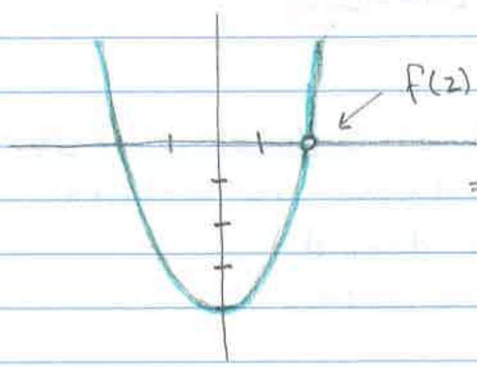
Ex: Find the values of x for which each function is continuous:

1) $f(x) = x^2 - 4$



$\Rightarrow$ all values of x: $(-\infty, \infty)$

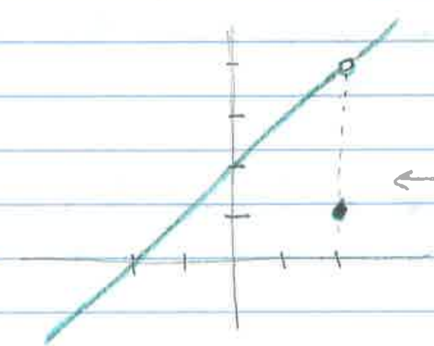
2) $f(x) = \frac{(x^2-4)(x-2)}{x-2}$



$f(2)$ is not defined

$\Rightarrow$ all values of $x \neq 2$: $(-\infty, 2) \cup (2, \infty)$

3) $f(x) = \begin{cases} x+2 & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$



$\Rightarrow (-\infty, 2) \cup (2, \infty)$

$f(2)$ is defined, and $\lim\limits_{x \to 2} f(x)$ exists, but

$\lim\limits_{x \to 2} f(x) \neq f(2)$

Properties of Continuous Functions

1) The constant function $f(x) = c$ is cts ("continuous") everywhere.
2) " identity " $f(x) = x$ " " ".

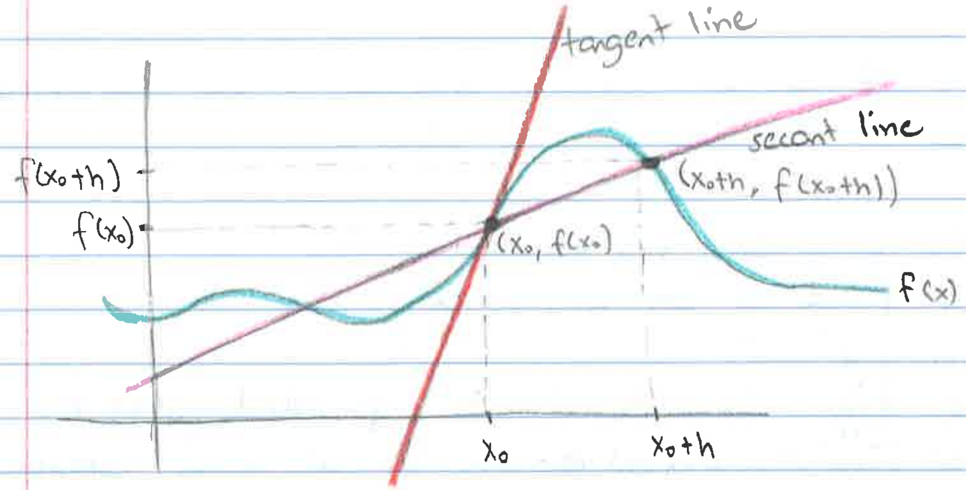
If f, g are cts at $x = a$, then
3) $f \pm g$ is cts at $x = a$
4) fg " " " "
5) f/g " " " " provided $g(a) \neq 0$
6) $[f(x)]^n$, n = real #, is cts at $x = a$ whenever it is defined there.

7) A polynomial function $y = P(x)$ is cts for all x
8) A rational function $R(x) = P(x)/q(x)$ is cts for all x where $q(x) \neq 0$.

Ex: Where is $R(x) = \dfrac{x^2 + 2x + 1}{x^2 - 4}$ cts?

$\Rightarrow$ cts everywhere except $x = 2$ and $x = -2$, by property 8.

2.6 Derivatives



- slope of secant line = "average rate of change of f over $[x_0, x_0+h]$"

$$m_{sec} = \frac{f(x_0+h) - f(x_0)}{x_0+h - x_0}$$

$$= \frac{f(x_0+h) - f(x_0)}{h}$$

- slope of tangent line = "instantaneous rate of change of f at x_0"

$$m_{tan} = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

Def. The derivative of a function f with respect to x is

$$f'(x) = \lim_{x\to 0} \frac{f(x+h) - f(x)}{h}$$

Ex: Use definition to find $f'(x)$.

1) $f(x) = x^2+1$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2hx + h^2 - \cancel{x^2}}{h}$$

$$= \lim_{h\to 0} \frac{2hx + h^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}}$$

$$= \lim_{h\to 0} 2x + \cancel{h}^{\to 0}$$

$$= 2x$$

2) $f(x) = 3/x$

$$f'(x) = \lim_{h\to 0} \frac{\frac{3}{x+h} - \frac{3}{x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{3x - 3(x+h)}{(x+h)x}}{h}$$

$$= \lim_{h\to 0} \frac{3x - 3x - 3h}{hx(x+h)}$$

$$= \lim_{h\to 0} \frac{-3\cancel{h}}{\cancel{h}x(x+h)}$$

$$= \lim_{h\to 0} \frac{-3}{x^2+xh}$$

$$= \frac{-3}{x^2}$$

Def. A function f is <u>differentiable</u> at x_0 if

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$$

exists.

<u>Theorem</u>: If a function is differentiable at x_0, then it is continuous at x_0.

Proof/ Assume f is differentiable at x_0.

Need to show: 1) $f(x_0)$ is defined

2) $\lim_{x \to x_0} f(x)$ exists

3) $\lim_{x \to x_0} f(x) = f(x_0)$

def. of continuity

Have: $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ exists

$$\left[\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}\right]\left[\lim_{h \to 0} h\right]^{\to 0} = 0$$

$$\Rightarrow \lim_{h \to 0} \frac{[f(x_0+h) - f(x_0)]\cancel{h}}{\cancel{h}} = 0$$

$$\Rightarrow \lim_{h \to 0} f(x_0+h) - f(x_0) = 0$$

$$\Rightarrow \lim_{h \to 0} f(x_0+h) = f(x_0)$$

let $x = x_0 + h$

$$\Rightarrow \lim_{x - x_0 \to 0} f(x) = f(x_0)$$

$$\Rightarrow \lim_{x \to x_0} f(x) = f(x_0). \checkmark$$

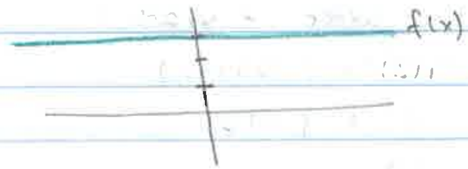
3.1 Basic Rules of Differentiation

Notation : $f'(x) = \frac{d}{dx}\left[f(x)\right]$

Rules

1) $\frac{d}{dx}(c) = 0$, c = constant

Ex: $f(x) = 3$



$\Rightarrow \frac{d}{dx} f(x) = \frac{d}{dx}(3) = 0$

* can also verify algebraically

2) Power Rule : $\frac{d}{dx}(x^r) = r x^{r-1}$, r = real #

Ex: $\frac{d}{dx}\left(x^{-\frac{1}{2}}\right) = -\frac{1}{2} x^{-\frac{1}{2}-1} = -\frac{1}{2} x^{-3/2}$

3) Constant Multiple Rule : $\frac{d}{dx}\left[c f(x)\right] = c \frac{d}{dx}\left[f(x)\right]$

Ex: $\frac{d}{dx}\left(\frac{\pi}{x}\right) = \frac{d}{dx}\left(\pi x^{-1}\right)$

$= \pi \frac{d}{dx}\left(x^{-1}\right)$

$= -\pi x^{-2}$

4) Sum and Difference Rule:

$\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right]$

Ex: $\frac{d}{dx}\left(4x^5 + 3x^2 - x\right) = \frac{d}{dx}\left(4x^5\right) + \frac{d}{dx}\left(3x^2\right) - \frac{d}{dx}(x)$

$= 20x^4 + 6x - 1$

Ex: The bat population over the next 10 years is given by

$P(t) = 3t^3 + 2t^2 - 10t + 600 \quad (0 \le \overset{\swarrow \text{years}}{t} \le 10)$

1) Find $P'(t)$:

$P'(t) = 9t^2 + 4t - 10$

2) What is the rate of change of the bat population after 2 years?

$P'(2) = 9(2)^2 + 4(2) - 10$

$= 36 + 8 - 10$

$= 34$

3) What is the bat population at that time?

$P(2) = 3(2)^3 + 2(2)^2 - 10(2) + 600$

$= 24 + 8 - 20 + 600$

$= 612$