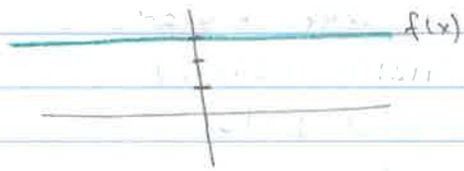
3.1 Basic Rules of Differentiation

Notation : $f'(x) = \frac{d}{dx}[f(x)]$

Rules

1) $\frac{d}{dx}(c) = 0$, c = constant

Ex: $f(x) = 3$



$\Rightarrow \frac{d}{dx} f(x) = \frac{d}{dx}(3) = 0$

* can also verify algebraically

2) Power Rule : $\frac{d}{dx}(x^r) = r x^{r-1}$, r = real #

Ex: $\frac{d}{dx}(x^{-\frac{1}{2}}) = -\frac{1}{2} x^{-\frac{1}{2}-1} = -\frac{1}{2} x^{-3/2}$

3) Constant Multiple Rule : $\frac{d}{dx}[c f(x)] = c \frac{d}{dx}[f(x)]$

Ex: $\frac{d}{dx}\left(\frac{\pi}{x}\right) = \frac{d}{dx}(\pi x^{-1})$

$= \pi \frac{d}{dx}(x^{-1})$

$= -\pi x^{-2}$

4) Sum and Difference Rule:

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

Ex: $\frac{d}{dx}(4x^5 + 3x^2 - x) = \frac{d}{dx}(4x^5) + \frac{d}{dx}(3x^2) - \frac{d}{dx}(x)$

$= 20x^4 + 6x - 1$

Ex: The bat population over the next 10 years is given by

$$P(t) = 3t^3 + 2t^2 - 10t + 600 \quad (0 \le \underset{\text{years}}{t} \le 10)$$

1) Find $P'(t)$:

$$P'(t) = 9t^2 + 4t - 10$$

2) What is the rate of change of the bat population after 2 years?

$$P'(2) = 9(2)^2 + 4(2) - 10$$
$$= 36 + 8 - 10$$
$$= 34$$

3) What is the bat population at that time?

$$P(2) = 3(2)^3 + 2(2)^2 - 10(2) + 600$$
$$= 24 + 8 - 20 + 600$$
$$= 612$$

3.2 The Product and Quotient Rules

Product Rule:

$$\frac{d}{dx}\left[f(x)g(x)\right] = f(x)g'(x) + f'(x)g(x)$$

⚠ $\frac{d}{dx}\left[f(x)g(x)\right] \neq f'(x)g'(x)$

Ex: $f(x) = x, \quad g(x) = 2x^2$

$$\frac{d}{dx}\left[f(x)g(x)\right] = \frac{d}{dx}\left[2x^3\right] = 6x^2$$

not equal!

$$f'(x)g'(x) = (1)(4x) = 4x$$

Ex: Find the derivative of

1) $f(x) = \underbrace{(3x^2 + x + 2)}_{p(x)}\underbrace{(x^3+1)}_{q(x)}$

$$f'(x) = p'(x)q(x) + p(x)q'(x)$$
$$= (6x+1)(x^3+1) + (3x^2+x+2)(3x^2)$$

2) $f(x) = x^3(\sqrt{x}+1)$

$$= \underbrace{x^3}_{p(x)}\underbrace{(x^{1/2}+1)}_{q(x)}$$

$$= p'(x)q(x) + p(x)q'(x)$$
$$= (3x^2)(x^{1/2}+1) + (x^3)(\tfrac{1}{2}x^{-1/2})$$

Ex: Suppose $g(x) = (x^3+1)f(x)$, where $f(1) = 3$ and $f'(1) = 0$. What is $g'(1)$?

$$g'(x) = (3x^2)f(x) + (x^3+1)f'(x)$$
$$\Rightarrow g'(1) = (3\cdot 1^2)\cdot f(1) + (1^3+1)f'(1)$$
$$= 3\cdot 3 + \cancel{2\cdot 0} \to 0$$
$$= 9$$

Quotient Rule

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2} \qquad (\text{if } g(x) \neq 0)$$

" Low d-high minus high d-low
over the square of what's below "

⚠ In general, $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] \neq \frac{f'(x)}{g'(x)}$

Ex: $\frac{d}{dx}\left(\frac{x^3}{x^2}\right) = \frac{d}{dx}(x) = 1$

not equal

$$\frac{d/dx\,(x^3)}{d/dx\,(x^2)} = \frac{3x^2}{2x} = \frac{3}{2}x$$

Ex: Find $f'(x)$:

1) $f(x) = \dfrac{x^2+1}{x^2-1}$ $\begin{matrix} \} \, p(x) \\ \} \, q(x) \end{matrix}$, $p'(x) = 2x$, $q'(x) = 2x$

$$f'(x) = \frac{p'(x)\,q(x) - p(x)\,q'(x)}{[q(x)]^2}$$

$$= \frac{(2x)(x^2-1) - (x^2+1)(2x)}{(x^2-1)^2}$$

$$= \frac{\cancel{2x^3} - 2x - \cancel{2x^3} - 2x}{(x^2-1)^2}$$

$$= \frac{-4x}{(x^2-1)^2}$$

2) $h(x) = \dfrac{\sqrt{x}}{x^2+1}$

$$h'(x) = \frac{\frac{1}{2}x^{-1/2}(x^2+1) - x^{1/2}(2x)}{(x^2+1)^2}$$

$$= \frac{\frac{1}{2}x^{-1/2}\left[x^2+1 - \overbrace{x(4x)}^{4x^2}\right]}{(x^2+1)^2}$$

$$= \frac{-3x^2+1}{2\sqrt{x}\,(x^2+1)^2}$$

3.3 The Chain Rule

Consider function $F(x) = (x^2+x+1)^{100}$. What is $F'(x)$?

Our strategy: Expand $F(x)$, then find the derivative of each monomial and add them.

$\Rightarrow$ Too long!

$\Rightarrow$ We need a new strategy.

Note: $F(x)$ is a composite function:

$f(x) = x^2+x+1$

$g(x) = x^{100}$

$\Rightarrow F(x) = g \circ f(x)$

Thought Process: We know that the derivative of each function is the rate of change of that function.
If $f(x)$ changes m times as fast as x and $g(x)$ changes n times as fast as x; that is

$f'(x) = m$ and $g'(x) = n$,

then $F(x)$ should change $m \cdot n$ times as fast as x;

$F'(x) = mn$.

The Chain Rule

If $h(x) = g[f(x)]$, then

$$h'(x) = \frac{d}{dx} g[f(x)] = g'[f(x)] \cdot f'(x)$$

Ex: Find the derivative:

1) $F(x) = (x^2 + x + 1)^{100}$

$\Rightarrow F(x) = g[f(x)]$, $\quad f(x) = x^2 + x + 1 \rightarrow f'(x) = 2x + 1$

$\quad g(x) = x^{100} \rightarrow g'(x) = 100x^{99}$

$$F'(x) = g'[f(x)] \cdot f'(x)$$
$$= 100(x^2 + x + 1)^{99} \cdot (2x + 1)$$

2) $G(x) = \sqrt{x^2 + 1}$

$$G'(x) = \underbrace{\tfrac{1}{2}(x^2+1)^{-1/2}}_{\text{derivative of outside}} \cdot \underbrace{(2x)}_{\text{derivative of inside}}$$

$$= \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}$$

$$= \frac{x}{\sqrt{x^2+1}}$$

3) $H(x) = \underbrace{(2x^2+3)^4}_{f(x)} \underbrace{(3x-1)^5}_{g(x)}$

$H'(x) \overset{\text{product rule}}{=} f'(x)\,g(x) + f(x)\,g'(x)$

where $f'(x) = 4(2x^2+3)^3 \cdot 4x = 16x(2x^2+3)^3$

$g'(x) = 5(3x-1)^4 \cdot 3 = 15(3x-1)^4$

$$\Rightarrow H'(x) = [16x(2x^2+3)^3][(3x-1)^5] + [(2x^2+3)^4][15(3x-1)^4]$$
$$= 16x(2x^2+3)^3(3x-1)^5 + 15(2x^2+3)^4(3x-1)^4$$
$$= (2x^2+3)^3(3x-1)^4\left[\underbrace{16x(3x-1)}_{48x^2-16x} + \underbrace{15(2x^2+3)}_{30x^2+45}\right]$$
$$= (2x^2+3)^3(3x-1)^4\left[78x^2 - 16x + 45\right]$$

Ex: Find the slope of the tangent line to the graph of the function

$$f(x) = \left(\frac{2x+1}{3x+2}\right)^3$$

at point $(0, 1/8)$

$$f'(x) = 3\left(\frac{2x+1}{3x+2}\right)^2 \cdot \underbrace{\frac{d}{dx}\left(\frac{2x+1}{3x+2}\right)}_{\frac{2(3x+2)-3(2x+1)}{(3x+2)^2}}$$

$$= 3\left(\frac{2x+1}{3x+2}\right)^2 \cdot \left[\frac{\overbrace{2(3x+2)}^{6x+4} - \overbrace{3(2x+1)}^{6x+3}}{1(3x+2)^2}\right]$$

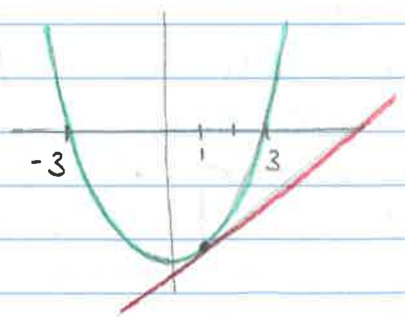
$$= 3\,\frac{(2x+1)^2}{(3x+2)^4}$$

$$f'(0) = \frac{3(2(0)+1)^2}{(3(0)+2)^4} = \frac{3}{2^4} = \frac{3}{16}$$

Ex: Find an equation of the tangent line to the graph of function

$$f(x) = x^2 - 9$$

at point $(1, -8)$.



slope $= f'(1)$, $f'(x) = 2x$

$= 2(1) = 2$

point-slope form:

$y - y_0 = m(x - x_0)$

$\Rightarrow y + 8 = 2(x-1)$