3.5 Higher Order Derivatives

* We can differentiate the derivative of a function!

Ex: $f(x) = 5x^3 - 3x^2 + 10x + 5$

first derivative:

$f'(x) = 15x^2 - 6x + 10$

second derivative:

$$f''(x) = \frac{d}{dx} f'(x) = 30x - 6$$

third derivative:

$$f'''(x) = \frac{d}{dx} f''(x) = 30$$

fourth derivative:

$$f^{(4)}(x) = \frac{d}{dx} f'''(x) = 0$$

Def. The nth derivative of a function $f(x)$ is

$$f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x),$$

where $f^{(n-1)}$ is the $(n-1)^{th}$ derivative.

Ex: The 2nd derivative of $f(x)$ is

$$f''(x) = f^{(2)}(x) = \frac{d}{dx} f'(x)$$

$$\Rightarrow f^{(2)}(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$$

Ex: 1) Find the 3rd derivative of $f(x) = x^{2/3}$.

1st: $f'(x) = \frac{2}{3} x^{2/3 - 1} = \frac{2}{3} x^{-1/3}$

2nd: $f''(x) = \frac{d}{dx} f'(x) = \frac{-1}{3} \cdot \frac{2}{3} x^{-1/3 - 1}$

$= \frac{-2}{9} x^{-4/3}$

3rd, $f'''(x) = \frac{d}{dx} f''(x) = \frac{-4}{3} \cdot \frac{-2}{9} x^{-4/3 - 1}$

$= \frac{8}{27} x^{-7/3}$

What is the domain of $f'''(x)$?

$(-\infty, 0) \cup (0, \infty)$

2) For what n will the nth derivative of

$p(x) = 3x^4 + 2x^2 + 1$

equal 0?

0th: $p(x) = 3x^4 + 2x^2 + 1$

1st: $p'(x) = 12x^3 + 4x$

2nd: $p''(x) = 36x^2 + 4$

3rd: $p'''(x) = 72x$

4th: $p^{(4)}(x) = 72$

5th: $p^{(5)}(x) = 0$

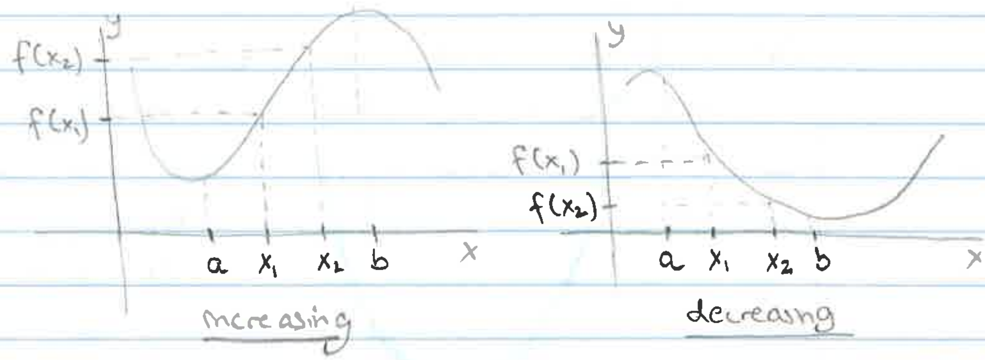
Fact - If $p(x)$ is a polynomial of degree n, then $p^{(k)}(x) = 0$ for $k \geq n+1$.

4.1 Applications of the First Derivative

Increasing and Decreasing Functions

Def. A function f is increasing (resp. decreasing) on an interval (a,b) if for all x_1, x_2 in (a,b), $f(x_1) < f(x_2)$ (resp. $f(x_1) > f(x_2)$) whenever $x_1 < x_2$.

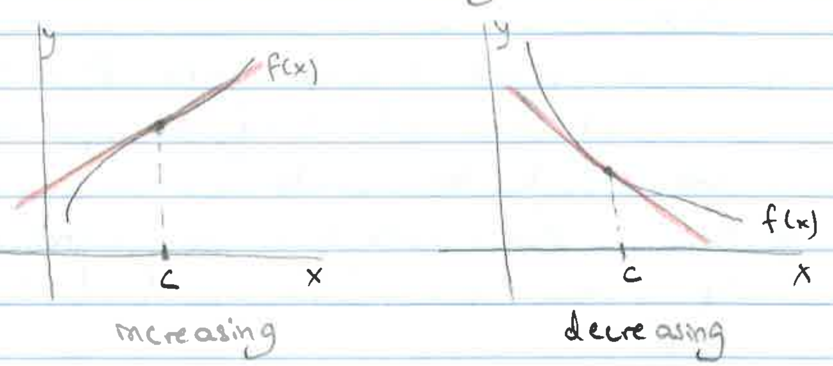
Ex:



Def. We say f is increasing (resp. decreasing) at a number c if there exists an interval (a,b) containing c s.t. f is increasing (resp. decreasing) on (a,b)

Fact - If the derivative at a point is positive, the slope of the tangent line at that point is positive, and the function is increasing.
- Likewise for decreasing.

Ex:

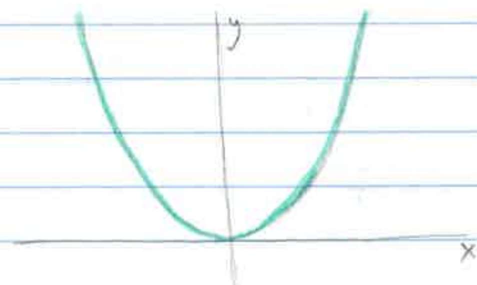


Theorem

1) If $f'(x) > 0$ (resp. $f'(x) < 0$) for every value of x in an interval (a,b), then f is increasing (resp. decreasing) on (a,b).

2) If $f'(x) = 0$ for every value of x in an interval (a,b), then f is constant on (a,b).

Ex: Find the interval where $f(x) = x^2$ is increasing and the interval where it is decreasing



increasing : $(0, \infty)$
decreasing : $(-\infty, 0)$

$f'(x) = 2x \Rightarrow f'(x) > 0$ for $x > 0$

$f'(x) < 0$ for $x < 0$

$\Rightarrow$ increasing at $(0, \infty)$

decreasing at $(-\infty, 0)$

Note: A continuous function cannot change sign unless it equals zero for some value of x

Determining Intervals Where a Function is Increasing or Decreasing

1) Find all values of x for which $f'(x) = 0$ or f' is discontinuous, and identify the open intervals determined by these numbers.

2) Select a test number c in each interval found in step 1, and determine the sign of $f'(c)$ in that interval.

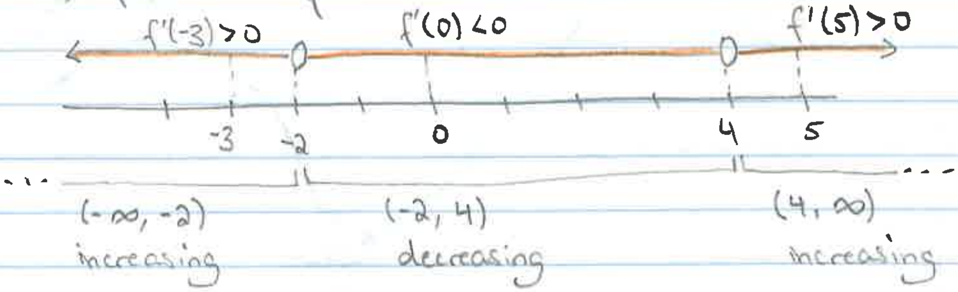
a. If $f'(c) > 0$, f is increasing on that interval

b. If $f'(c) < 0$, f is decreasing on that interval

Ex: Determine the intervals where
$f(x) = x^3 - 3x^2 - 24x + 32$
is increasing, and where it is decreasing.

$$f'(x) = 3x^2 - 6x - 24$$
$$= 3(x^2 - 2x - 8)$$
$$= 3(x-4)(x+2)$$

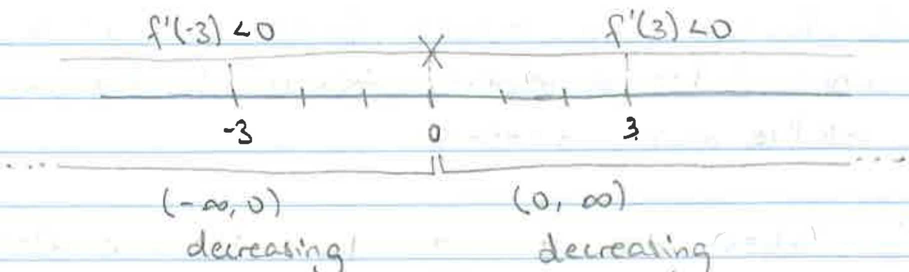
$\Rightarrow f'(x) = 0$ for $x = 4$ and $x = -2$

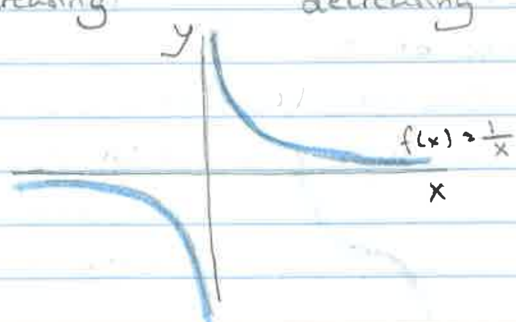


Ex: Determine the intervals where $f(x) = \frac{1}{x}$
is increasing and where it is decreasing.

$$f'(x) = -x^{-2} = \frac{-1}{x^2}$$

$\Rightarrow f'(x)$ undefined for $x = 0$

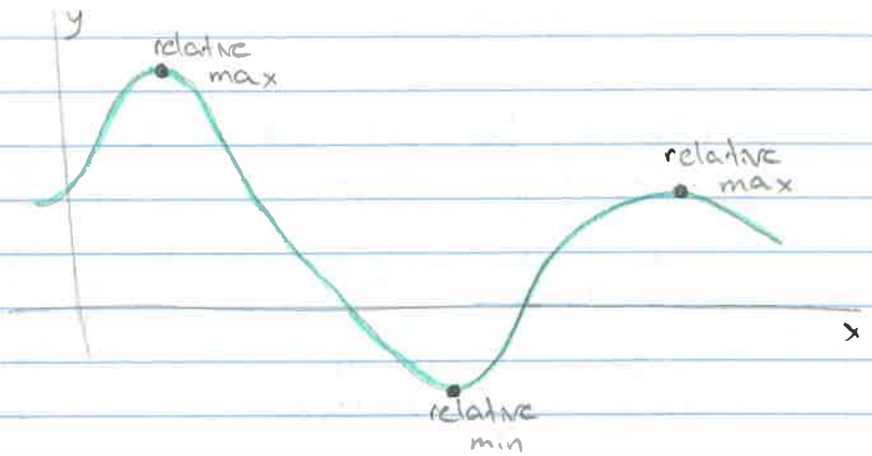




Relative Extrema

Def. The relative maxima and relative minima of a function correspond to the "high points" and "low points" of a function.

Ex:



Def. A function f has a relative maximum (resp. relative minimum) at $x = c$ if there exists an open interval (a,b) containing c s.t. $f(x) \le f(c)$ (resp. $f(x) \ge f(c)$) for all x in (a,b).

Note: If f is a differentiable function, then at any c where f has a relative extremum (relative min or relative max), $f'(c) = 0$.

⚠ $f'(c) = 0$ does not imply that f has a relative extrema at c.

Ex: Consider $f(x) = x^3$

$f'(0) = 0$ but f does not have a relative extremum at 0.