Ex: Determine the relative extrema of the function

$f(x) = x^3 - 3x^2 - 24x + 32$

$f'(x) = 3x^2 - 6x - 24$

$= 3(x^2 - 2x - 8)$

$= 3(x-4)(x+2)$

$\Rightarrow f'(x) = 0$ for $x = 4$ and $x = -2$

$f''(x) = 6x - 6$

$f''(4) = 6(4) - 6 = 24 - 6 = 18 > 0$

$\rightarrow$ relative min at $x = 4$

$f''(-2) = 6(-2) - 6 = -18 < 0$

$\rightarrow$ relative max at $x = -2$

4.3 Curve Sketching

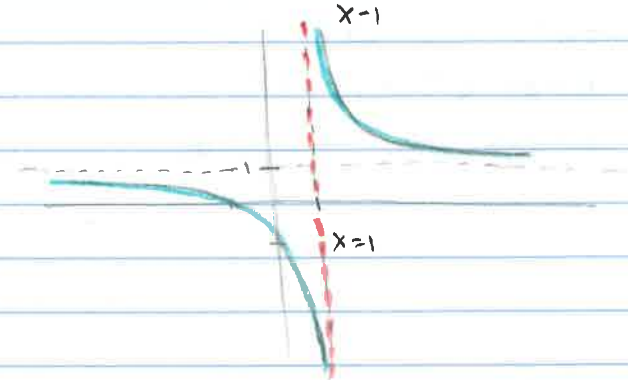
Vertical Asymptotes

Def. The line $x=a$ is a vertical asymptote of f if

$$\lim_{x\to a^+} f(x) = \infty \text{ or } -\infty$$

$$\lim_{x\to a^-} f(x) = \infty \text{ or } -\infty$$

Ex: Consider $f(x) = \dfrac{x+1}{x-1}$



$$\lim_{x\to 1^+} f(x) = \infty$$

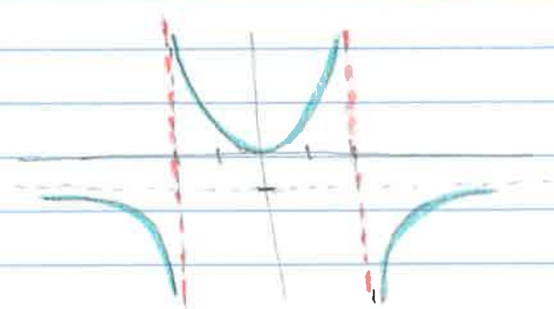
$$\lim_{x\to 1^-} f(x) = -\infty$$

Proposition - If $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function ($P(x)$ and $Q(x)$ are both polynomials), then line $x=a$ is a vertical asymptote of f if $Q(a)=0$ but $P(a)\neq 0$.

Ex: Find the vertical asymptotes of $f(x) = \dfrac{x^2}{4-x^2}$

Want to find the zeros of $Q(x) = 4-x^2$:

$$4-x^2 = 0 \iff (2-x)(2+x) = 0$$
$$\iff x=2 \text{ or } x=-2$$

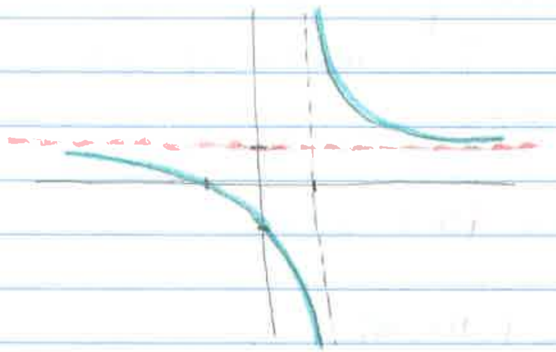


Horizontal Asymptotes

Def. The line $y = b$ is a horizontal asymptote of f if

$$\lim_{x \to \infty} f(x) = b \quad \text{or} \quad \lim_{x \to -\infty} f(x) = b$$

Ex: Consider $f(x) = \dfrac{x+1}{x-1}$



$$\lim_{x \to \infty} f(x) = \lim_{x \to -\infty} f(x) = 1$$

Ex: Find the horizontal asymptotes of $f(x) = \dfrac{x^2}{4-x^2}$.

$$\lim_{x \to \infty} \frac{x^2}{4-x^2} = -1 \quad \text{and} \quad \lim_{x \to -\infty} \frac{x^2}{4-x^2} = -1.$$

horizontal asymptote: $y = -1$.

A Guide to Curve Sketching

1) Determine the domain of f.
2) Find the x- and y- intercepts.
3) Find all horizontal and vertical asymptotes of f.
4) Find the relative extrema of f
5) Determine the intervals where f is increasing and where f is decreasing.
6) Find the inflection points of f
7) Determine where f is concave up and where f is concave down.

Example: Sketch $f(x) = \dfrac{3(x+2)}{(x-1)^2}$

1) Domain: $(-\infty, 1) \cup (1, \infty)$

2) y-int = 6 $\quad (f(0) = 6)$

x-int = -2 $\quad (f(-2) = 0)$

3) horizontal asymptotes: $\lim\limits_{x\to\infty} \dfrac{3(x+2)}{(x-1)^2} = \lim\limits_{x\to\infty} \dfrac{3x}{3x^2} = 0$

$\lim\limits_{x\to-\infty} \dfrac{3(x+2)}{(x-1)^2} = 0$

vertical asymptotes: $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^+} f(x) = \infty$

4) & 5) extrema:

$$f'(x) = \frac{3(x-1)^2 - 3(x+2)\cdot 2(x-1)}{(x-1)^{\cancel{4}3}}$$

$$= \frac{3(x-1) - 6(x+2)}{(x-1)^3}$$

$$= \frac{-3(x+5)}{(x-1)^3}$$

$\Rightarrow$ critical pts: $x = 1$, $x = -5$

f': $\quad - \quad$ (-5) $\quad + \quad$ (1) $\quad -$

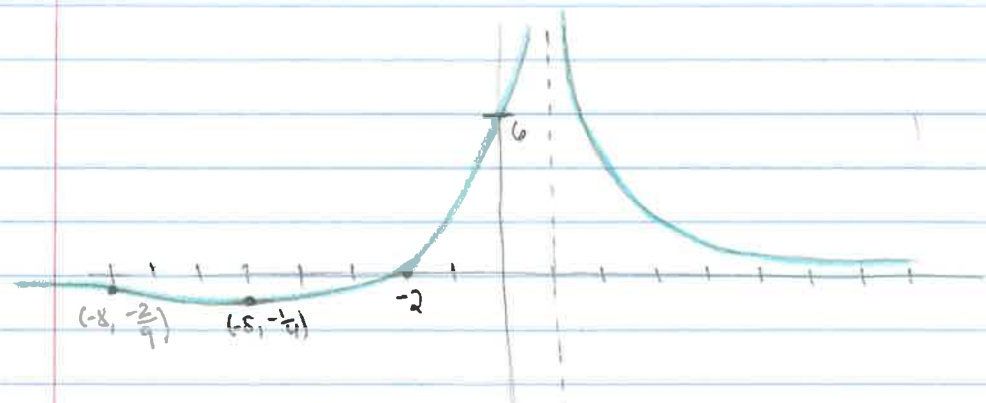
rel min: $(-5, -\frac{1}{4})$

6) 27) concavity:

$$f''(x) = \frac{6(x+8)}{(x-1)^4}$$

f'' — -8 + 1 +

inflection pt: $(-8, -2/9)$



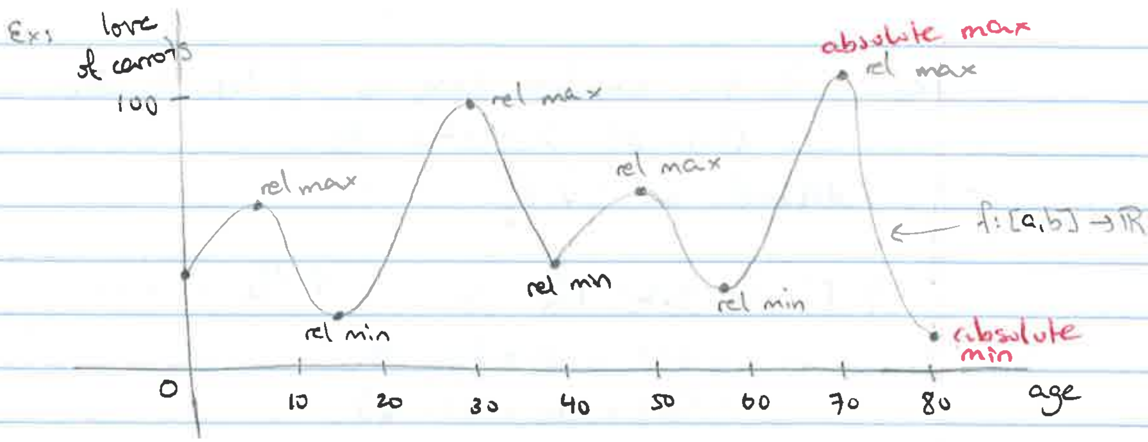
4.5 Optimization I

* We look for the largest or smallest value a function can take.

Absolute Extrema

Def. If $\left\{\begin{array}{l} f(x) \le f(c) \\ f(x) \ge f(c) \end{array}\right\}$ for all x in the domain of f, then $f(c)$ is called the $\left\{\begin{array}{l} \text{absolute maximum value} \\ \text{absolute minimum value} \end{array}\right\}$ of f

Ex:

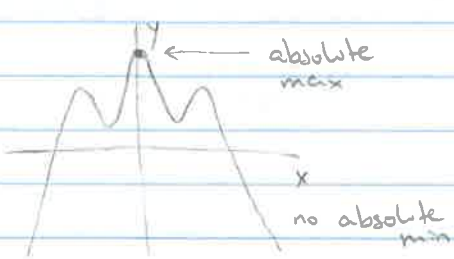


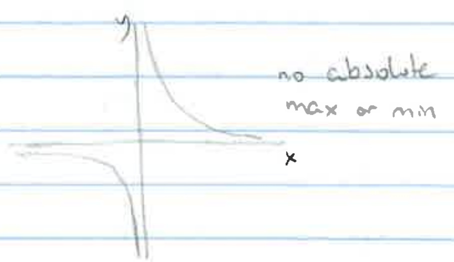
Absolute Extrema on a Closed Interval

Theorem (Extreme Value Theorem) - If a function f is continuous on a closed interval $[a,b]$, then f has absolute maximum and an absolute minimum on $[a,b]$.

Note: A cts function defined on an arbitrary interval does not always have an absolute max or min.

Ex:





<u>Note</u>: The absolute extremum of a cts function f on $[a,b]$ may occur at one or both endpoints of $[a,b]$.

<u>Finding Absolute Extrema of f on a Closed Interval</u>

1) Find all critical numbers of f that lie in (a,b).
2) Compute the value of f at each critical number found in Step 1 and compute $f(a)$ and $f(b)$.
3) The absolute max and absolute min value of f will correspond to the largest and smallest numbers, resp., found in Step 2.

Ex: Find the absolute extrema of the function

$$f(x) = x^3 - 2x^2 - 4x + 4$$

defined on $[0,3]$

1) $f'(x) = 3x^2 - 4x - 4$ P: -12, S: -4 $\}$ $-6, +2$

$$= 3x^2 - 6x + 2x - 4$$
$$= 3x(x-2) + 2(x-2)$$
$$= (3x+2)(x-2)$$

$\Rightarrow f'(x) = 0$ when $3x+2 = 0$ or $x - 2 = 0$ $\Rightarrow$ $x = -2/3$ or $x = 2$

2) Evaluate $f(x)$ for $x = 0, 2, 3$ ($-2/3$ is outside desired interval)

x	0	2	3
$f(x)$	$\boxed{4}$	$2^3 - 2(2^2) - 4(2) + 4$ $= \boxed{-4}$	$3^3 - 2(3^2) - 4(3) + 4$ $= 27 - 18 - 12 + 4 = \boxed{1}$

3) absolute max: $(0,4)$ absolute min: $(2,-4)$

Ex: Find the absolute max and min values of
$f(x) = x^{2/3}$ on $[-1, 8]$

$f'(x) = \frac{2}{3} x^{-1/3}$ → crit. pt. $x = 0$

x	-1	0	8
$f(x)$	1	0	$(\sqrt[3]{8})^2 = 4$

absolute min: $(0,0)$

absolute max: $(8,4)$

Ex: Acrosonic's total profit from manufacturing and selling x units of their model F loudspeaker systems is given by

$P(x) = -0.02x^2 + 300x - 200{,}000$ $(0 \le x \le 20{,}000)$

How many units of the loudspeaker system must Acrosonic produce to maximize its profits?

$P'(x) = -0.04x + 300$
$\Rightarrow P'(x) = 0$ at $x = 7500$

x	0	7500	$20{,}000$
$P(x)$	$-200{,}000$	$P(7500) = 925{,}000$	$P(20{,}000) = -2{,}200{,}000$

⇒ Acrosonic will realize its maximum profit by producing 7500 units.

4.5 Optimization II

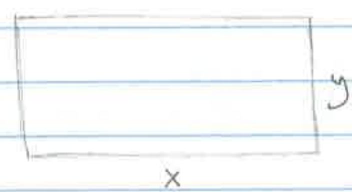
We consider problems in which we are first required to find the appropriate function to be optimized.

Guidelines for Solving Optimization Problems

1) Assign a letter to each variable mentioned in the problem.
2) Find an expression for the quantity to be optimized.
3) Use conditions given in the problem to write the quantity to be optimized as a function of one variable.
4) Optimize the function f over its domain.

Ex: A man wishes to have a rectangular shaped garden in his backyard. He has 50 feet of fencing with which to enclose his garden. Find the dimensions of the largest garden he can have if he uses all of the fencing.

1) Let x and y denote the dimensions (in feet) of the two adjacent sides of the garden



2) The area of the garden $A = xy$ is the quantity to be maximized.

3) The perimeter of the rectangle, $2x + 2y$, must equal 50 feet:

$$2x + 2y = 50$$
$$\Rightarrow \quad y = 25 - x$$
$$\Rightarrow \quad A = x(25 - x)$$
$$= -x^2 + 25x$$

- Since the sides of the rectangle must be ≥ 0, we must have $x \geq 0$ and $y = 25 - x \geq 0$. That is, $0 \leq x \leq 25$.

$\Rightarrow$ Problem is reduced to finding the absolute maximum of

$$A = -x^2 + 25x$$

on the closed interval $[0, 25]$

5) $A' = -2x + 25$

$\Rightarrow A'(x) = 0$ for $x = 25/2 = 12.5$

x	0	12.5	25
$A(x)$	0	156.25	0

- Since $y = 25 - x$, $y = 12.5$ when $x = 12.5$.

$\Rightarrow$ The garden of maximum area ($156.25\ ft^2$) is a square with sides of length 12.5 ft.

Ex: Betty Moore Company requires that its containers have a capacity of 54 in^3, have the shape of a cylinder, and be made of aluminum. Determine the radius and height of the container that requires the least amount of metal.

Let r = radius of container
h = height of container
S = surface area of container

Note: The amount of aluminum used to construct the container is given by the total surface area of the cylinder, S.
$\Rightarrow$ S is the quantity to be minimized.

$$S = 2\pi r^2 + 2\pi r h$$

Since the volume of the container must be 54 in^3:

$$\pi r^2 h = 54$$

$$\Rightarrow \quad h = \frac{54}{\pi r^2}$$

$$S = 2\pi r^2 + 2\pi r\left(\frac{54}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{108}{r}$$

Optimization: $S' = 4\pi r - \dfrac{108}{r^2}$