Ex: Find the absolute max and min values of
$f(x) = x^{2/3}$ on $[-1, 8]$

$f'(x) = \frac{2}{3} x^{-1/3}$ → crit. pt. $x = 0$

x	-1	0	8
$f(x)$	1	0	$(\sqrt[3]{8})^2 = 4$

absolute min: $(0,0)$ absolute max: $(8,4)$

Ex: Acrosonic's total profit from manufacturing and selling x units of their model F loudspeaker systems is given by
$$P(x) = -0.02x^2 + 300x - 200{,}000 \qquad (0 \le x \le 20{,}000)$$
How many units of the loudspeaker system must Acrosonic produce to maximize its profits?

$P'(x) = -0.04x + 300$
$\Rightarrow P'(x) = 0$ at $x = 7500$

x	0	7500	$20{,}000$
$P(x)$	$-200{,}000$	$P(7500) = 925{,}000$	$P(20{,}000) = -2{,}200{,}000$

⇒ Acrosonic will realize its maximum profit by producing 7500 units.

4.5 Optimization II

We consider problems in which we are first required to find the appropriate function to be optimized.

Guidelines for Solving Optimization Problems

1) Assign a letter to each variable mentioned in the problem.
2) Find an expression for the quantity to be optimized.
3) Use conditions given in the problem to write the quantity to be optimized as a function of one variable.
4) Determine the domain
4) Optimize the function f over its domain.

Ex: A man wishes to have a rectangular shaped garden in his backyard. He has 50 feet of fencing with which to enclose his garden. Find the dimensions of the largest garden he can have if he uses all of the fencing.

1) Let x and y denote the dimensions (in feet) of the two adjacent sides of the garden

2) The area of the garden $A = xy$ is the quantity to be maximized.

3) The perimeter of the rectangle, $2x + 2y$, must equal 50 feet:

$$2x + 2y = 50$$
$$\Rightarrow y = 25 - x$$
$$\Rightarrow A = x(25 - x)$$
$$= -x^2 + 25x$$

4) • Since the sides of the rectangle must be ≥ 0,
we must have $x \geq 0$ and $y = 25 - x \geq 0$.
That is, $0 \leq x \leq 25$.

$\Rightarrow$ Problem is reduced to finding the absolute maximum of

$$A = -x^2 + 25x$$

on the closed interval $[0, 25]$

5) $A' = -2x + 25$
$\Rightarrow A'(x) = 0$ for $x = 25/2 = 12.5$

x	0	12.5	25
$A(x)$	0	156.25	0

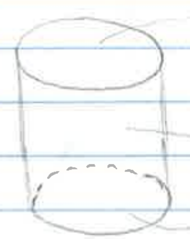
• Since $y = 25 - x$, $y = 12.5$ when $x = 12.5$.

$\Rightarrow$ The garden of maximum area ($156.25\ ft^2$) is a square with sides of length 12.5 ft.

Ex: Betty Moore Company requires that its containers have a capacity of 54 m^3, have the shape of a cylinder, and be made of aluminum. Determine the radius and height of the container that requires the least amount of metal.

Let r = radius of container
h = height of container
S = surface area of container

<u>Note</u>: The amount of aluminum used to construct the container is given by the total surface area of the cylinder, S.
$\Rightarrow$ S is the quantity to be minimized.



$$S = 2\pi r^2 + 2\pi r h$$

Since the volume of the container must be 54 m^3:

volume of cylinder $= \pi r^2 h = 54$

$$\Rightarrow \quad h = \frac{54}{\pi r^2}$$

$$S = 2\pi r^2 + 2\pi r\left(\frac{54}{\pi r^2}\right)$$

$$= 2\pi r^2 + \frac{108}{r}$$

Optimization: $S' = 4\pi r - \dfrac{108}{r^2}$

finding critical points $S' = 0 \implies 4\pi r = \frac{108}{r^2}$

$$r^3 = \frac{108}{4\pi} = \frac{27}{\pi}$$

$$r = \sqrt[3]{\frac{27}{\pi}} = \frac{3}{\sqrt[3]{\pi}}$$

Now we show that this value of r gives rise to an absolute minimum:

Need to show S is concave up at $r = 3\pi^{-1/3}$.

$$S'' = 4\pi + \frac{216}{r^3}$$

$$\implies S''\Big|_{r=3\pi^{-1/3}} = 4\pi + \frac{216}{r^3}\Big|_{r=3\pi^{-1/3}} > 0$$

$\implies S$ is concave up at $r = 3\pi^{-1/3}$

$\implies (3\pi^{-1/3}, S(3\pi^{-1/3}))$ is a relative min of S.

absolute min:

Note that $S'' > 0$ for all $r > 0$, so S is always concave upward.

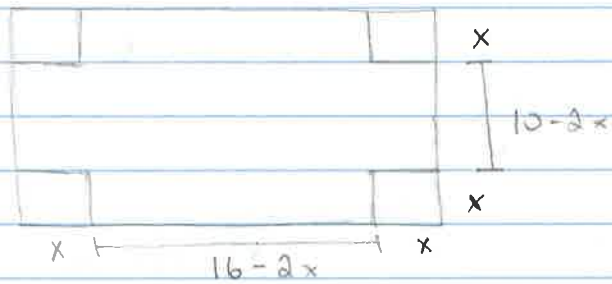
$\implies (3\pi^{-1/3}, S(3\pi^{-1/3}))$ is an absolute min of S.

$$h = \frac{54}{\pi r^2} = \frac{54}{\pi(3\pi^{-1/3})^2}$$

$$= \frac{54\pi^{2/3}}{\pi(9)} = \frac{6}{\pi^{1/3}}$$

We conclude that the required container has radius $3\pi^{-1/3}$ in and height $6\pi^{-1/3}$ in.

Ex: By cutting away identical squares from each corner of a rectangular piece of cardboard and folding up the resulting flaps, the cardboard may be turned into an open box. If the cardboard is 16 inches long and 10 inches wide, find the dimensions of the box that will yield the maximum volume.



$\Rightarrow V = (16-2x)(10-2x)x$

$= 4(x^3 - 13x^2 + 40x)$

quantity to be maximized

• Since each side must be nonnegative, x must satisfy the inequalities

$x \geq 0$

$16 - 2x \geq 0 \Rightarrow 16 \geq 2x \Rightarrow x \leq 8$

$10 - 2x \geq 0 \Rightarrow 10 \geq 2x \Rightarrow x \leq 5$

$\Rightarrow 0 \leq x \leq 5$

$V' = 4(3x^2 - 26x + 40)$ $\left.\begin{matrix} P: 120 \\ S: -26 \end{matrix}\right\} \Rightarrow -20, -6$

$= 4(3x^2 - 20x - 6x + 40)$

$= 4[x(3x-20) - 2(3x-20)]$

$= 4(x-2)(3x-20)$

$\Rightarrow V' = 0$ for $x = 2$, $20/3$

outside of $[0,5]$, so we can ignore it

x	0	2	5
$V(x)$	0	$4(2^3-13(2^2)+40(2))$ $= 144$	0

$\Rightarrow$ Volume of the box is maximized by taking $x=2$

$\Rightarrow$ Dimensions of desired box is $12'' \times 6'' \times 2''$, and its volume is 144 in^3.

5.1 Exponential Functions

Def. The function defined by

$$f(x) = b^x \quad (b > 0,\ b \neq 1)$$

is called an exponential function with base b and exponent x.
The domain of f is the set of all real #s.

Ex: $f(x) = 2^x$

Laws of Exponents

Let $a, b > 0$, and let x, y be real #s. Then,

1) $b^x \cdot b^y = b^{x+y}$
2) $\frac{b^x}{b^y} = b^{x-y}$
3) $(b^x)^y = b^{xy}$
4) $(ab)^x = a^x b^x$
5) $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

Ex: Let $f(x) = 2^{2x-1}$. Find the value of x for which $f(x) = 16$

$16 = 2^{2x-1}$

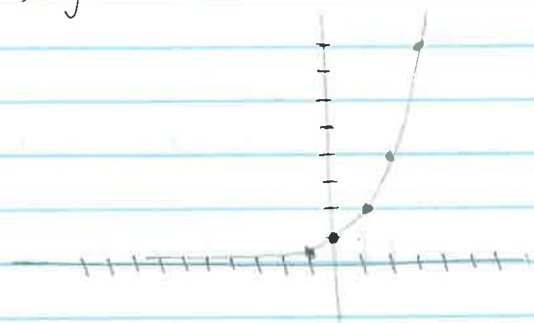
$\Rightarrow 2^4 = 2^{2x-1}$

$\Rightarrow 4 = 2x - 1$

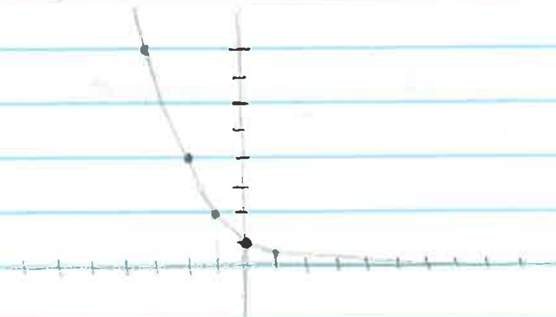
$\Rightarrow x = 5/2$

Ex: Sketch the exponential function

1) $y = 2^x$



2) $y = (\frac{1}{2})^x$



Note - $y = b^x$ is an $\left\{\begin{matrix} \text{increasing} \\ \text{decreasing} \\ \text{constant} \end{matrix}\right\}$ function of x

if $\left\{\begin{matrix} b > 1 \\ 0 < b < 1 \\ b = 1 \end{matrix}\right\}$.

Properties of the Exponential Function ($y = b^x$, $b > 0$, $b \neq 1$)

1) Domain = $(-\infty, \infty)$
2) Range = $(0, \infty)$
3) It passes through point $(0, 1)$
4) It is cts on $(-\infty, \infty)$
5) It is increasing on $(-\infty, \infty)$ if $b > 1$ and decreasing on $(-\infty, \infty)$ if $b < 1$.