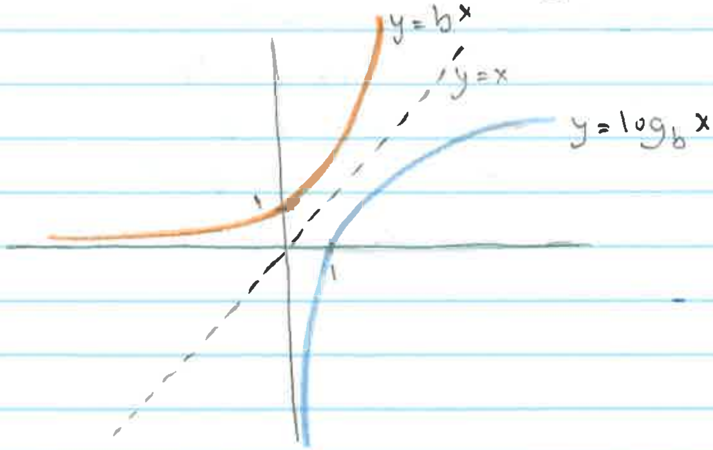
Logarithmic Functions and Their Graphs

Def. The function defined by $f(x) = \log_b x$ $(b>0,\ b \neq 1)$ is called the logarithmic function with base b.
The domain of f is the set of all positive numbers.

* The graph of $y = \log_b x$ is the reflection across the $y = x$ line of the graph of $y = b^x$:



* In general, inverse functions are reflections of each other across the $y = x$ line.

Properties of Logarithmic Function $(y = \log_b x,\ b>0,\ b \neq 1)$

1) Domain = $(0, \infty)$
2) Range = $(-\infty, \infty)$
3) Its graph passes through point $(1, 0)$.
4) It is cts on $(0, \infty)$
5) It is increasing on $(0, \infty)$ if $b>1$ and decreasing on $(0, \infty)$ if $b<1$.

Properties Relating e^x and $\ln x$

$e^{\ln x} = x$ for $x>0$

$\ln e^x = x$ for any real number x

Ex: Solve the equation $2e^{x+5} = 5$.

$$2e^{x+5} = 5 \implies e^{x+5} = 5/2$$
$$\implies \ln(e^{x+5}) = \ln(5/2)$$
$$\implies x+5 = \ln(5/2)$$
$$\implies x = \ln(5/2) - 5$$

Ex: Solve the equation $5\ln x + 3 = 0$.

$$5\ln x + 3 \implies 5\ln x = -3$$
$$\implies \ln x = -3/5$$
$$\implies e^{\ln x} = e^{-3/5}$$
$$\implies x = e^{-3/5}$$

5.4 Differentiation of Exponential Functions

Derivative of the Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

Ex: Find the derivative of the functions

1) $f(x) = x^2 e^x$

$$f'(x) = \frac{d}{dx}(x^2)\, e^x + x^2 \frac{d}{dx}(e^x)$$

$$= 2xe^x + x^2 e^x$$

$$= xe^x(2+x)$$

2) $f(x) = (e^x + 2)^{3/2}$

$$f'(x) = \frac{3}{2}(e^x+2)^{1/2} e^x$$

The Chain Rule for Exponential Functions

If $f(x)$ is a differentiable function, then

$$\frac{d}{dx}(e^{f(x)}) = e^{f(x)} \cdot f'(x)$$

HW: Prove this rule (using Chain Rule).

Ex: Find the derivative of the functions.

1) $f(x) = e^{2x}$

$$f'(x) = e^{2x} \cdot 2 = 2e^{2x}$$

2) $g(t) = e^{2t^2+t}$

$$g'(t) = e^{2t^2+t} \cdot (4t+1) = (4t+1)\, e^{2t^2+t}$$

3) $y = x e^{-2x}$

$$\frac{dy}{dx} = \frac{d}{dx}(x)\, e^{-2x} + x \frac{d}{dx}\left(e^{-2x}\right)$$

$$= e^{-2x} + x\, e^{-2x} \cdot (-2)$$

$$= e^{-2x}(1 - 2x)$$

4) $g(t) = \dfrac{e^t}{e^t + e^{-t}}$

$$g'(t) = \frac{\frac{d}{dt}(e^t)\,(e^t + e^{-t}) - e^t \frac{d}{dt}\left(e^t + e^{-t}\right)}{(e^t + e^{-t})^2}$$

$$= \frac{e^t(e^t + e^{-t}) - e^t(e^t - e^{-t})}{(e^t + e^{-t})^2}$$

$$= \frac{e^{2t} + 1 - (e^{2t} - 1)}{(e^t + e^{-t})^2}$$

$$= \frac{2}{(e^t + e^{-t})^2}$$

5.5 Differentiation of Logarithmic Functions

Derivative of $\ln x$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad (x \neq 0)$$

Proof/ $f(x) = \ln x$

$$\Rightarrow e^{f(x)} = e^{\ln x} = x$$

$$\Rightarrow \frac{d}{dx}\left(e^{f(x)}\right) = \frac{d}{dx}(x)$$

$$\Rightarrow e^{f(x)} \cdot f'(x) = 1$$

$$\Rightarrow f'(x) = \frac{1}{e^{f(x)}} = \frac{1}{x}$$

Ex: Find the derivative of the function.

1) $f(x) = x\ln x$

$$f'(x) = \frac{d}{dx}(x)\ln x + x\frac{d}{dx}(\ln x)$$

$$= \ln x + \frac{x}{x} = \ln x + 1$$

2) $g(x) = \dfrac{\ln x}{x}$

$$g'(x) = \frac{\frac{1}{x}(x) - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

Chain Rule and Logarithmic Functions

If $f(x)$ is a differentiable function, then

$$\frac{d}{dx}\left[\ln f(x)\right] = \frac{f'(x)}{f(x)} \qquad (f(x) > 0)$$

HW: Prove this rule (using Chain Rule).

Ex: Find the derivative of the function.

1) $f(x) = \ln(x^2+1)$

$$f'(x) = \frac{d/dx\,(x^2+1)}{x^2+1} = \frac{2x}{x^2+1}$$

2) $y = \ln[(x^2+1)(x^3+2)^6]$

$$= \ln(x^2+1) + \ln[(x^3+2)^6]$$

$$= \ln(x^2+1) + 6\ln(x^3+2)$$

$$\frac{dy}{dx} = \frac{d/dx\,(x^2+1)}{x^2+1} + 6\,\frac{d/dx\,(x^3+2)}{x^3+2}$$

$$= \frac{2x}{x^2+1} + \frac{18x^2}{x^3+2}$$

3) $g(t) = \ln(t^2 e^{-t^2})$

$$= \ln(t^2) + \ln(e^{-t^2})$$

$$= 2\ln t + -t^2\cancel{\ln e}^{\,1}$$

$$= 2\ln t - t^2$$

$$g'(t) = \frac{2}{t} - 2t$$

Logarithmic Differentiation

Finding the derivative of a function can sometimes be made easier by applying the laws of logarithms to simplify the function.

This process is called logarithmic differentiation.

Ex: Differentiate the function using logarithmic differentiation

1) $y = x(x+1)(x^2+1)$

$$\Rightarrow \ln y = \ln[x(x+1)(x^2+1)]$$

$$= \ln x + \ln(x+1) + \ln(x^2+1)$$

$$\Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}\left[\ln x + \ln(x+1) + \ln(x^2+1)\right]$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(\ln x) + \frac{d}{dx}\ln(x+1) + \frac{d}{dx}\ln(x^2+1)$$

$$= \frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}\right)$$

$$= x(x+1)(x^2+1)\left(\frac{1}{x} + \frac{1}{x+1} + \frac{2x}{x^2+1}\right)$$

<u>Finding dy/dx by Logarithmic Differentiation</u>

1) Take the natural logarithm on both sides of the equation, and use properties of logarithms to write any complicated expression as a sum of simpler terms.
2) Differentiate both sides of the equation w.r.t. x
3) Solve the resulting equation for $\frac{dy}{dx}$.

Ex: 2) $y = x^2(x-1)(x^2+4)^3$

$$\Rightarrow \ln y = \ln\left[x^2(x-1)(x^2+4)^3\right]$$

$$= \ln(x^2) + \ln(x-1) + \ln\left[(x^2+2)^3\right]$$

$$= 2\ln x + \ln(x-1) + 3\ln(x^2+2)$$

$$\Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}\left(2\ln x + \ln(x-1) + 3\ln(x^2+2)\right)$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+2}$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+2}\right)$$

$$= x^2(x-1)(x^2+4)^3\left(\frac{2}{x} + \frac{1}{x-1} + \frac{6x}{x^2+2}\right)$$