5.3 Compound Interest

Def. Given a sum of money P (called the principal), the simple interest I is the interest computed on P:

$$I = Prt$$

principal; rate per year; # of years

Def. The accumulated amount A is the sum of the principal and interest after t years:

$$\begin{aligned} A &= P + I \\ &= P + Prt \\ &= P(1 + rt) \end{aligned}$$

Def. The interest earned is periodically added to the principal and thereafter earns interest itself at the same rate.

This is called the compound interest.

Ex: Suppose \$1000 (the principal) is deposited in the bank for a term of 3 years, earning interest at the rate of 8% per year (called the nominal rate) compounded annually. Then

$$\begin{aligned} A_1 &= P(1 + rt) \\ &= 1000(1 + 0.08(1)), \quad P = 1000, \quad r = 0.08, \quad t = 1 \\ &= 1000 + 80 \\ &= 1080 \end{aligned}$$

$$\begin{aligned} A_2 &= P_1(1 + rt), \quad P_1 = A_1 \\ &= A_1(1 + rt) \\ &= 1080(1 + 0.08(1)) \\ &= 1000(1 + 0.08(1))^2, \quad \text{since } 1080 = 1000(1 + 0.08(1)) \end{aligned}$$

$$A_3 = P_2(1+rt), \quad P_2 = A_2$$
$$= A_2(1+rt)$$
$$\approx 1166.40(1+0.08(1))$$
$$= 1000\,(1+0.08(1))^3$$
$$\approx 1259.71$$

<u>Remark</u>: $A_1 = P(1+rt)$
$A_2 = P(1+rt)^2$
$A_3 = P(1+rt)^3$
$\Rightarrow A_n = P(1+rt)^n$

This formula is derived under the assumption that interest is compounded annually. However, the interest is usually compounded more than once a year.

Def. The interval of time between successive interest calculations is called the <u>conversion period</u>.

$i = \dfrac{r}{m}$ annual interest rate / periods per year

Ex: If $r = 0.08$ and interest is compounded quarterly ($m = 4$), then

$$i = \frac{0.08}{4} = 0.02 \Rightarrow 2\% \text{ per period.}$$

$\Rightarrow$ 1st period: $A_1 = P(1+i)$
2nd period: $A_2 = A_1(1+i) = P(1+i)^2$
3rd period: $A_3 = P(1+i)^3$
$\vdots$
nth period: $A_n = P(1+i)^n$

Compound Interest Formula

$$A = P\left(1 + \frac{r}{m}\right)^{mt}, \quad \text{where}$$

A = accumulated amount at the end of t years
P = principal
r = nominal interest rate per year
m = # of conversion periods per year
t = term (# of years).

Ex: Find the accumulated amount after 3 years if \$1000 is invested at 8% per year compounded

1) annually: $P = 1000, \quad r = 0.08, \quad m = 1, \quad t = 3$

$$\Rightarrow A = 1000(1 + 0.08)^3$$
$$\approx 1259.71$$

2) semiannually: $P = 1000, \quad r = 0.08, \quad m = 2, \quad t = 3$

$$\Rightarrow A = 1000\left(1 + \frac{0.08}{2}\right)^{(2)(3)}$$

$$= 1000(1 + 0.04)^6$$
$$\approx 1265.32$$

3) monthly: $P = 1000, \quad r = 0.08, \quad m = 12, \quad t = 3$

$$\Rightarrow A = 1000\left(1 + \frac{0.08}{12}\right)^{(12)(3)}$$

$$\approx 1270.24$$

4) daily: $P = 1000, \quad r = 0.08, \quad m = 365, \quad t = 3$

$$\Rightarrow A = 1000\left(1 + \frac{0.08}{365}\right)^{(365)(3)}$$

$$\approx 1271.22$$

Effective Rate of Interest

* The interest earned on an investment depends on the frequency with which the interest is compounded.
  ⇒ The yearly interest rate does not reflect the rate at which interest is earned.

Def. The effective rate is the simple interest rate that would produce the same accumulated amount in 1 year as the nominal rate compounded m times a year.
Also called the true rate, or annual percentage yield

Ex: Assume an initial investment of P dollars. Then the accumulated amount after 1 year at a simple interest rate of r_{eff} (effective rate) per year is

$$A = P(1 + r_{eff})$$
$$= P\left(1 + \frac{r}{m}\right)^m \quad , \text{ since } t = 1$$

$$\Rightarrow 1 + r_{eff} = \left(1 + \frac{r}{m}\right)^n$$

Effective Rate of Interest Formula

$$r_{eff} = \left(1 + \frac{r}{m}\right)^m - 1 \quad , \text{ where}$$

r_{eff} = effective rate of interest
r = nominal interest rate per year
m = # of conversion periods per year.

Ex: Find the effective rate of interest corresponding to a nominal rate of 8% per year compounded

1) annually ($m=1$) : $r_{eff} = (1+0.08) - 1 = 0.08$

2) quarterly ($m=4$) : $r_{eff} = \left(1+\frac{0.08}{4}\right)^4 - 1 \approx 0.0824$

3) monthly ($m=12$) : $r_{eff} = \left(1+\frac{0.08}{12}\right)^{12} - 1 \approx 0.0830$

4) daily ($m=365$) : $r_{eff} = \left(1+\frac{0.08}{365}\right)^{365} - 1 \approx 0.0833$

Present Value

Compound interest formula: $A = P\left(1+\frac{r}{m}\right)^{mt}$

Def. The principal P is often referred to as the present value.
The accumulated value A is called the future value.

Present Value Formula for Compound Interest

$$P = A\left(1+\frac{r}{m}\right)^{-mt}$$

Ex: How much money should be deposited in a bank paying interest at the rate of 6% per year compounded monthly so that at the end of 3 years, the accumulated amount will be \$20,000?

$A = 20000, \quad r = 0.06, \quad m = 12, \quad t = 3$

$$\Rightarrow P = 20000\left(1+\frac{0.06}{12}\right)^{-(12)(3)}$$

Continuous Compounding of Interest

* What happens to the accumulated amount over a fixed period of time if the interest is compounded more and more frequently?

$$A = P\left(1+\frac{r}{m}\right)^{mt} \rightsquigarrow \text{want } m \to \infty$$

$$\lim_{m\to\infty} P\left(1+\frac{r}{m}\right)^{mt} = P\Big[\underbrace{\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{m}}_{e^r}\Big]^{t}$$

$$= Pe^{rt}$$

Continuous Compound Interest Formula

$A = Pe^{rt}$ where

P = principal

r = annual interest rate compounded continuously

t = time in years

A = accumulated amount at the end of t years.

Ex: Find the accumulated amount after 3 years if 1000 is invested at 8% per year compounded

1) daily, 2) continuously

1) $A = 1000\left(1+\frac{0.08}{365}\right)^{365(3)} \approx 1271.22$

2) $A = 1000\, e^{(0.08)(3)} \approx 1271.25$

Using Logarithms to Solve Problems Involving Compound Interest

Ex: How long will it take \$10,000 to grow to \$15,000 if the investment earns an interest rate of 12% per year compounded quarterly?

$A = 15000, \quad P = 10{,}000, \quad r = 0.12, \quad m = 4, \quad t = ?$

$\Rightarrow \quad 15000 = 10000\left(1 + \dfrac{0.12}{4}\right)^{4t}$

$\Rightarrow \quad \dfrac{15000}{100\cancel{00}} = (1 + 0.03)^{4t}$

$\Rightarrow \quad \dfrac{3}{2} = (1.03)^{4t}$

$\Rightarrow \quad \ln(3/2) = \ln\left[(1.03)^{4t}\right]$

$= 4t \ln(1.03)$

$\Rightarrow \quad t = \dfrac{\ln(3/2)}{4\ln(1.03)} \approx 3.43$

$\Rightarrow$ It will take approximately 3.4 years for the investment to grow from \$10,000 to \$15,000

Ex: Find the interest rate needed for an investment of \$10,000 to grow to \$18,000 in 5 years if the interest is compounded monthly.

$A = 18,000, \quad P = 10,000, \quad m = 12, \quad t = 5, \quad r = ?$

$$\Rightarrow \quad 18000 = 10000\left(1 + \frac{r}{12}\right)^{(12)(5)}$$

$$\Rightarrow \quad \frac{18000}{10000} = \left(1 + \frac{r}{12}\right)^{60}$$

$$\Rightarrow \quad \underbrace{\frac{9}{5}}_{=} = \left(1 + \frac{r}{12}\right)^{6}$$

$$\Rightarrow \quad \ln(1.8) = \ln\left[(1 + r/12)^6\right] = 6\ln(1 + r/12)$$

$$\Rightarrow \quad \ln(1 + r/12) = \frac{\ln(1.8)}{6}$$

$$\Rightarrow \quad e^{\ln(1 + r/12)} = e^{\ln(1.8)/6}$$

$$\Rightarrow \quad 1 + r/12 = e^{\ln(1.8)/6}$$

$$\Rightarrow \quad r/12 = e^{\ln(1.8)/6} - 1$$

$$\Rightarrow \quad r = 12\left(e^{\ln(1.8)/6} - 1\right) \approx 0.1181$$

5.6 Exponential Functions as Mathematical Models.

Exponential Growth

Recall: $f(x) = b^x$ is an increasing function when $b > 1$;
in particular $f(x) = e^x$ is an increasing function.

Consider: $Q(t) = Q_0 e^{kt}$
$\uparrow$ a quantity at time t

This function has the following properties:

1) $Q(0) = Q_0$
2) $Q(t)$ increases w/o bound as t increases w/o bound.

3) $$\begin{aligned} Q'(t) &= d/dt\,(Q_0 e^{kt}) \\ &= Q_0\, d/dt\,(e^{kt}) \\ &= kQ_0 e^{kt} \\ &= k\,Q(t) \end{aligned}$$

$\Rightarrow$ As $Q(t)$ increases, so does the rate of increase of $Q(t)$.

$\Rightarrow$ The exponential function $Q(t) = Q_0 e^{kt}$ provides us w/ a mathematical model of a quantity $Q(t)$ that is initially present in the amount $Q(0) = Q_0$ and whose rate of growth at time t is directly proportional to the amount of quantity present at time t.

Def. Such a quantity is said to exhibit unrestricted <u>exponential growth</u>, and the constant k of proportionality is called the <u>growth constant</u>.

Ex: Interest earned on a fixed deposit when compounded continuously exhibits exponential growth.

Ex: (Growth of Bacteria) The # of bacteria in a culture grows in accordance w/ the law

$Q(t) = Q_0 e^{kt}$, where

Q_0 = # of bacteria initially present in the culture

k = growth constant (determined by strain of bacteria under consideration, etc)

t = time elapsed (in hours)

Suppose 10000 bacteria a present initially and 60000 present 2 hours later.

1) How many bacteria will be in the culture at the end of 4 hours?

$60000 = 10000\, e^{k(2)}$

$\Rightarrow e^{k(2)} = 6$

$\Rightarrow \ln(e^{k(2)}) = \ln(6)$

$\Rightarrow 2k = \ln(6)$

$\Rightarrow k = \ln(6)/2$

$\Rightarrow Q(t) = 10000\, e^{\ln(6)/2\, t}$

$\Rightarrow Q(4) = 1000\, e^{2\ln(6)}$

2) What is the rate of growth of the population after 4 hours?

$Q(t) = Q_0 e^{kt}$

rate of growth $\rightarrow Q'(t) = k Q_0 e^{kt}$

$\Rightarrow Q'(4) = k Q_0 e^{k(4)} = k\, Q(4)$

$= \frac{\ln(6)}{2} \cdot 10000\, e^{2\ln(6)}$ $\quad [e^{\ln(6)}]^2 = 6^2 = 36$

$= 5000 \ln(6)\, e^{2\ln(6)}$

$= 180000 \ln(6)$

$$\begin{array}{r} 36 \\ \times\ 5 \\ \hline 180 \end{array}$$