5.6 Exponential Functions as Mathematical Models.

Exponential Growth

Recall: $f(x) = b^x$ is an increasing function when $b > 1$;
in particular $f(x) = e^x$ is an increasing function.

Consider: $Q(t) = Q_0 e^{kt}$

$\uparrow$ a quantity at time t

This function has the following properties:

1) $Q(0) = Q_0$
2) $Q(t)$ increases w/o bound as t increases w/o bound.
3) $Q'(t) = d/dt\,(Q_0 e^{kt})$
$= Q_0\, d/dt\,(e^{kt})$
$= kQ_0 e^{kt}$
$= k\,Q(t)$

$\Rightarrow$ As $Q(t)$ increases, so does the rate of increase of $Q(t)$.

$\Rightarrow$ The exponential function $Q(t) = Q_0 e^{kt}$ provides us w/ a mathematical model of a quantity $Q(t)$ that is initially present in the amount $Q(0) = Q_0$ and whose rate of growth at time t is directly proportional to the amount of quantity present at time t.

Def. Such a quantity is said to exhibit unrestricted <u>exponential growth</u>, and the constant k of proportionality is called the <u>growth constant</u>.

Ex: Interest earned on a fixed deposit when compounded continuously exhibits exponential growth: $A = Pe^{rt}$, $P, r > 0$

Ex: (Growth of Bacteria) The # of bacteria in a culture grows in accordance w/ the law

$$Q(t) = Q_0 e^{kt}, \text{ where}$$

Q_0 = # of bacteria initially present in the culture

k = growth constant (determined by strain of bacteria under consideration, etc)

t = time elapsed (in hours)

Suppose 10000 bacteria a present initally and 60000 present 2 hours later.

1) How many bacteria will be in the culture at the end of 4 hours?

$$60000 = 10000\, e^{k(2)}$$
$$\Rightarrow e^{k(2)} = 6$$
$$\Rightarrow \ln(e^{k(2)}) = \ln(6)$$
$$\Rightarrow 2k = \ln(6)$$
$$\Rightarrow k = \ln(6)/2$$
$$\Rightarrow Q(t) = 10000\, e^{\ln(6)/2\, t}$$
$$\Rightarrow Q(4) = 1000\, e^{2\ln(6)}$$

2) What is the rate of growth of the population after 4 hours?

$$Q(t) = Q_0 e^{kt}$$

rate of growth $\rightarrow$ $Q'(t) = k Q_0 e^{kt}$

$$\Rightarrow Q'(4) = k Q_0 e^{k(4)} = k\, Q(4)$$
$$= \frac{\ln(6)}{2} \cdot 10000\, e^{2\ln(6)}$$

$[e^{\ln(6)}]^2 = 6^2 = 36$

$$= 5000 \ln(6)\, e^{2\ln(6)}$$
$$= 180000 \ln(6)$$

$$\begin{array}{r} 36 \\ 5 \\ \hline 180 \end{array}$$

Exponential Decay

A quantity exhibits exponential decay if it decreases at a rate that is directly proportional to its size:

$$Q(t) = Q_0 e^{-kt} \quad (0 \leq t < \infty), \text{ where}$$

Q_0 = amount present initially

k = positive number called the decay constant

$$\Rightarrow Q'(t) = -k Q_0 e^{-kt}$$
$$= -k\, Q(t)$$

Ex: (Radioactive Decay) - Radioactive substances decay exponentially.

Def. The half-life of a radioactive substance is the time required for a given amount to be reduced by half.

The half-life of Radium is ≈ 1600 years. Suppose there are initially 200 mg of radium. Find the amount left after t years.

$$Q(0) = Q_0 = 200$$
$$\Rightarrow Q(t) = 200\, e^{-kt}$$
$$Q(1600) = Q(0)/2 = 100 \quad \text{(given half-life)}$$
$$\Rightarrow 200\, e^{-k(1600)} = 100$$
$$\Rightarrow e^{-1600k} = 1/2$$
$$\Rightarrow \ln(e^{-1600k}) = \ln(1/2)$$
$$\Rightarrow -1600k = \ln(1/2)$$
$$\Rightarrow k = \frac{\ln(1/2)}{-1600} = \frac{\ln(2^{-1})}{-1600} = \frac{-\ln(2)}{-1600} = \frac{\ln(2)}{1600}$$
$$\Rightarrow Q(t) = 200\, e^{-\ln(2)/1600\; t}$$

What is the amount left after 800 years?

$$Q(800) = 200\, e^{-\ln(2)/1600 \cdot (800)}$$
$$= 200\, e^{-\ln(2)/2}$$

Ex: (Carbon-14 Dating) Carbon-14, a radioactive isotope of carbon, has a half-life of 5730 years. What is its decay constant?

$$Q(t) = Q_0 e^{-kt}$$

$$Q(5730) = Q_0 e^{-k(5730)} = Q_0/2$$

$$\Rightarrow e^{-5730k} = 1/2$$

$$\Rightarrow \ln e^{-5730k} = \ln(1/2) = \ln(2^{-1}) = -\ln(2)$$

$$\Rightarrow -5730k = -\ln(2)$$

$$\Rightarrow k = \frac{\ln(2)}{5730}$$

Ex: A skull from an archeological site has one-tenth the amount of C-14 that it originally contained. What is the age of the skull?

$$Q(t) = Q_0 e^{-kt}$$

$$= Q_0 e^{-\ln(2)/5730\, t}$$

Given: $Q_0 e^{-\ln(2)/5730\, t} = Q_0/10$

$$\Rightarrow e^{-\ln(2)/5730\, t} = 1/10$$

$$\Rightarrow -\ln(2)/5730\, t = \ln(1/10) = \ln(10^{-1}) = -\ln(10)$$

$$\Rightarrow t = -\ln(10) \cdot \frac{5730}{-\ln(2)}$$

$$= 5730 \frac{\ln(10)}{\ln(2)}$$

$$\approx 19,030$$

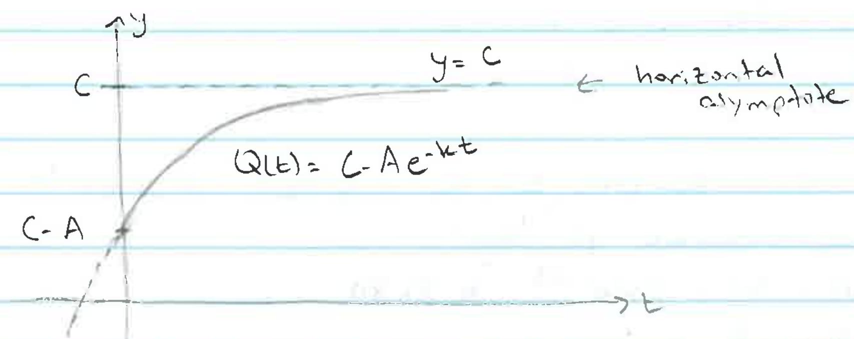
Learning Curves

Consider the function $Q(t) = C - Ae^{-kt}$, where $C, A, k > 0$.

$\rightarrow Q(0) = C - A$

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} C - Ae^{-kt}$$

$$= C - \lim_{t\to\infty} \underbrace{Ae^{-kt}}_{\to 0} = C$$



Note: Starting at $t=0$, $Q(t)$ increases rapidly, but then the rate of increase slows down considerably after a while.

$$\lim_{t\to\infty} Q'(t) = \lim_{t\to\infty} kAe^{-kt} = 0$$

* The behavior of Q resembles the learning pattern of workers engaged in repetitive work.

Def The graph of $Q(t) = C - Ae^{-kt}$ is often called a learning curve.

Ex: After completing a basic training program, a new, previously inexperienced employee will be able to assemble

$$Q(t) = 50 - 30e^{-0.5t}$$

cameras per day t months after the employee starts work on the assembly line.

1) How many cameras can a new employee assemble per day after basic training?

$$Q(0) = 50 - 30 = 20$$

2) How many cameras can an employee w/ 1 month of experience assemble per day?

$$Q(1) = 50 - 30e^{-0.5} \approx 31.80$$

Six months of experience?

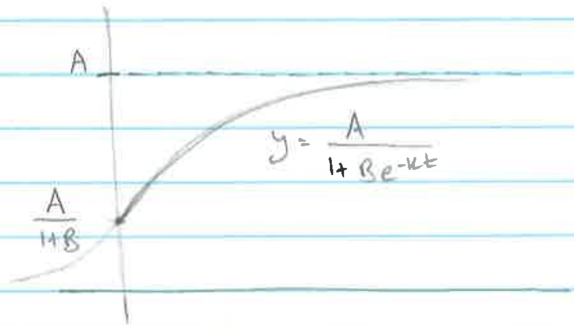
$$Q(6) = 50 - 30e^{-3} \approx 48.51$$

3) How many cameras can the average experienced employee assemble per day?

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} 50 - 30e^{-0.5t} = 50$$

Logistic Growth Function

Def. Logistic growth function: $Q(t) = \dfrac{A}{1+Be^{-kt}}$, A = carrying capacity of environment



• Logistic growth function models population growth if one considers growth constraints such as space, resources, etc.

Ex: The number of people who contracted the flu after t days during a flu epidemic is given by

$$Q(t) = \frac{5000}{1+1249e^{-kt}}.$$

If 40 people contracted the flu by day 7, how many contract the flu by day 15?

$$Q(7) = \frac{5000}{1+1249e^{-kt}} = 40$$

$$\Rightarrow 1+1249e^{-k(7)} = \frac{5000}{40} = \frac{500}{4} = 125$$

$$\Rightarrow e^{-7k} = \frac{125-1}{1249} = \frac{124}{1249}$$

$$\Rightarrow -7k = \ln\left(\frac{124}{1249}\right)$$

$$\Rightarrow k = -\frac{1}{7}\ln\left(\frac{124}{1249}\right) \approx 0.33$$

$$Q(t) = \frac{5000}{1+1249e^{-0.33t}}$$

$$Q(15) = \frac{5000}{1+1249e^{-0.33(15)}} \approx 508$$