6.1 Antiderivatives and the Rules of Integration

Goal: "Undo" differentiation

Def. A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

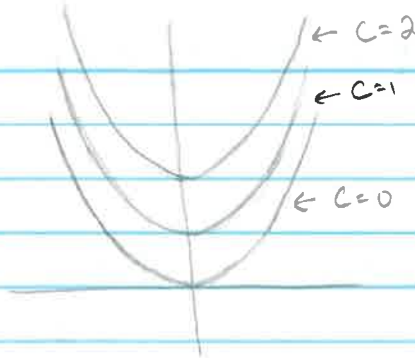
Ex: 1) $F(x) = x^2$ is an antiderivative of $f(x) = 2x$:
$F'(x) = 2x = f(x)$.

2) $F(x) = \frac{1}{3}x^3 - 2x^2 + x - 1$ is an antiderivative of $f(x) = x^2 - 4x + 1$:
$F'(x) = x^2 - 4x + 1 = f(x)$.

Ex: Let $F(x) = x$, $G(x) = x + 8$, $H(x) = x + C$ ← real #.
Then $F(x)$, $G(x)$, $H(x)$ are all antiderivatives of the function $f(x) = 1$:
$F'(x) = 1$, $G'(x) = 1$, $H'(x) = 1$
$\Rightarrow F'(x) = G'(x) = H'(x) = f(x)$.

Theorem - Let G be an antiderivative of function f on interval I. Then, every antiderivative F of f must be of the form
$F(x) = G(x) + C$,
where C is a constant.

Ex: Since $G(x) = x^2$ is an antiderivative of f, a general expression for antiderivatives of f is
$F(x) = G(x) + C$
$= x^2 + C$



Indefinite Integral

The process of finding all antiderivatives of a function is called antidifferentiation or integration.

Notation: $\int f(x)\,dx = F(x) + C$ ← constant of integration

integral sign, integrand, indefinite integral

Ex: $\int f(x)\,dx = x^2 + C$

Basic Integration Rules

1) Indefinite integral of a constant:

$\int k\,dx = kx + C$ (k, a constant)

Ex: $\int 2\pi\,dx = 2\pi x + C$

2) The Power Rule:

$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$ $(n \neq -1)$

Proof/ $\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{n+1}{n+1}x^{(n+1)-1} = x^n$

Ex: $\int \frac{1}{x^{3/2}}\,dx = \int x^{-3/2}\,dx$

$= \frac{1}{-3/2+1}x^{-3/2+1} + C = -2x^{-1/2} + C$

$\rightarrow \frac{-3}{2} + \frac{2}{2} = \frac{-1}{2}$

3) The indefinite integral of a constant multiple of a function:

$$\int c f(x)\,dx = c\int f(x)\,dx \qquad (c, \text{ a constant})$$

⚠ Only a constant can be moved out of an integral sign.

Ex: $\int x^2\,dx \neq x\int x\,dx$

Ex: $\int 4t^4\,dx = 4\int t^4\,dx = 4\cdot\frac{1}{5}t^5 + C$

4) The sum rule:

$$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

Ex: $\int 3x^2 + 4x^{-1/2} + x\,dx$

$$= \int 3x^2\,dx + \int 4x^{-1/2}\,dx + \int x\,dx$$

$$= (x^3 + C_1) + \left(4\cdot\frac{1}{1/2}x^{1/2} + C_2\right) + \left(\frac{x^2}{2} + C_3\right)$$

$$= x^3 + 8x^{1/2} + \frac{x^2}{2} + C, \qquad (C = C_1 + C_2 + C_3)$$

5) Indefinite integral of an exponential function:

$$\int e^x\,dx = e^x + C$$

Proof/ $\frac{d}{dx}(e^x + C) = e^x$

6) Indefinite integral of $f(x) = \frac{1}{x}$:

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

↑ absolute value

Proof/ $\frac{d}{dx}(\ln|x| + C) = \frac{1}{x}$

Ex: $\int 2x + \frac{3}{x} + \frac{4}{x^2}\, dx = \int 2x\, dx + \int \frac{3}{x}\, dx + \int \frac{4}{x^2}\, dx$

$$= 2\int x\, dx + 3\int \frac{1}{x}\, dx + 4\int x^{-2}\, dx$$

$$= 2\frac{x^2}{2} + 3\ln|x| + 4\frac{x^{-1}}{-1} + C$$

$$= x^2 + 3\ln|x| - \frac{4}{x} + C$$

Differential Equations

Suppose we are given

$$f'(x) = 2x - 1 \quad (*)$$

and we want to find $f(x)$.

We can find $f(x)$ by integrating both sides of the equation:

$$f(x) = \int f'(x)\, dx = \int (2x-1)\, dx = x^2 - x + C \quad (**)$$

Def. Equation $(*)$ is an example of a differential equation. In general, a differential equation is an equation that involves the derivative of an unknown function.

A solution of a differential equation is any function that satisfies the differential equation.

Equation $(**)$ is the general solution of the differential equation $f'(x) = 2x - 1$

- Though the differential equation has infinitely many solutions, we can obtain a particular solution by specifying the value the function must assume at a certain value of x.

Ex: Suppose f must satisfy $f(1) = 3$. Then
$$f(x)\Big|_{x=1} = (x^2 - x + C)\Big|_{x=1} = 3$$

$\Rightarrow (1)^2 - (1) + C = 3$

$\Rightarrow C = 3$

$\Rightarrow$ The particular solution is $f(x) = x^2 - x + 3$

Def. The condition $f(1) = 3$ is an example of an <u>initial condition</u>. More generally, an initial condition is a condition imposed on the value of f at some $x = a$.

<u>Initial Value Problems</u>

Def. An <u>initial value problem</u> is one in which we must find a function satisfying
(1) a differential equation
(2) one or more initial conditions.

Ex: Find function f satisfying
$f'(x) = 3x^2 - 4x + 8$ and $f(1) = 9$.

$$f(x) = \int f'(x)\,dx = \int (3x^2 - 4x + 8)\,dx$$
$$= x^3 - 2x^2 + 8x + C \quad \leftarrow \text{general solution}$$

$f(1) = 9 \Rightarrow (1)^3 - 2(1)^2 + 8(1) + C = 9$

$\Rightarrow 1 - 2 + 8 + C = 9$

$\Rightarrow C + 7 = 9 \quad \Rightarrow C = 2$

$\Rightarrow f(x) = x^3 - 2x^2 + 8x + 2 \quad \leftarrow$ particular solution