3.4 Marginal Functions in Economics

Def. Marginal analysis is the study of the rate of change of economic quantities.

Ex: An economist is not just concerned with an economy's gross domestic product (GDP), but also with the rate at which it is growing or decreasing.

Cost Functions

Ex: The total cost (in dollars) incurred each week by Apple for manufacturing x i-phones is given by cost function

$$C(x) = 8000 + 200x - 0.2x^2 \qquad (0 \le x \le 400)$$

a) What is the cost incurred for manufacturing the 251st i-phone?

$$\begin{aligned}\text{cost of making 251st i-phone} &= \left(\text{cost of making 251 i-phones}\right) - \left(\text{cost of making 250 i-phones}\right)\\ &= \boxed{C(251) - C(250)}\\ &= 8000 + 200(251) - 0.2(251)^2\\ &\quad - \left(8000 + 200(250) - 0.2(250)^2\right)\\ &= 99.8\end{aligned}$$

b) Find the rate of change of the total cost function wrt (with respect to) x when x = 250.

$$\begin{aligned}&= \boxed{C'(250)}\\ &= 200 - 0.4x \Big|_{x=250} \leftarrow \text{evaluated at}\\ &= 200 - 0.4(250)\\ &= 100\end{aligned}$$

(c) Compare the results obtained in (a) and (b).

(a) : 99.80

(b) : 100.00

Why are these numbers so close?

Because $C(251) - C(250) = \dfrac{C(251) - C(250)}{1}$

slope of secant line through points $(250, C(250))$, $(251, C(251))$ $\rightarrow$ $= \dfrac{C(250+1) - C(250)}{1}$

$$= \left.\frac{C(250+h) - C(250)}{h}\right|_{h=1}$$

$$\approx \left.\frac{C(250+h) - C(250)}{h}\right|_{h \text{ small}}$$

slope of tangent line at point $(250, C(250))$ $\rightarrow$ $\approx \lim_{h \to 0} \dfrac{C(250+h) - C(250)}{h}$

$$= C'(250)$$

Def. <u>Marginal cost</u> = The cost incurred in producing an additional unit of a product, given that a plant is already at a certain level of operation.

Ex: Cost of producing 251st i-phone in previous example

<u>Note</u>: We have seen that the marginal cost is approximated by rate of change of total cost function evaluated at appropriate point:

Ex: Cost of producing 251st i-phone $= C(251) - C(250) \approx C'(250)$

Def. <u>Marginal cost function</u> = derivative of total cost function

Ex: Bob's Bakery makes gluten-free cookies. The daily total cost of making these cookies is given by

$$C(x) = 0.0001x^3 - 0.08x^2 + 40x + 5000$$

where x is the number of cookies baked.

(a) Find the marginal cost function.

$$C'(x) = 0.0003x^2 - 0.16x + 40$$

(b) What is the marginal cost when $x = 200, 300, 400$, and 600?

$$C'(200) = 0.0003(200)^2 - 0.16(200) + 40$$
$$= 12 - 32 + 40$$
$$= 20$$

$$C'(300) = 0.0003(300)^2 - 0.16(300) + 40$$
$$= 27 - 48 + 40$$
$$= 19$$

$$C'(400) = 0.0003(400)^2 - 0.16(400) + 40$$
$$= 48 - 64 + 40$$
$$= 24$$

$$C'(600) = 0.0003(600)^2 - 0.16(600) + 40$$
$$= 108 - 96 + 40$$
$$= 52$$

(c) Interpret your results.

The cost of making the 201st cookie is approximately \$20. The cost of making the 301st and 401st cookies is a similar value. However, when the level of production is already at 600 units, producing one additional unit is approximately \$52.

The higher cost for producing the 601st unit may be due to: ① overtime costs
② production breakdown due to stress on the equipment
③ etc.

Average Cost Functions

Def Let $C(x)$ be the cost incurred in producing x units of some product. The average cost of producing x units of the product is

$$\underbrace{\overline{C}(x) = \frac{C(x)}{x}}_{\text{average cost function}}$$

Def. Marginal average cost function $= \frac{d}{dx}\left[\overline{C}(x)\right]$

Ex: The total cost (in dollars) of producing x units of a certain commodity is given by

$C(x) = 400 + 20x$

(a) Find the average cost function.

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{400}{x} + \frac{20\cancel{x}}{\cancel{x}} = \frac{400}{x} + 20$$

(b) Find the marginal average cost function.

$$\bar{C}'(x) = \frac{d}{dx}\left[\bar{C}(x)\right]$$

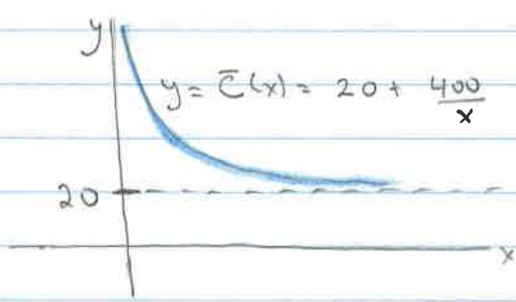
$$= \frac{d}{dx}\left(\frac{400}{x} + 20\right)$$

$$= \frac{-400}{x^2}$$

(c) What are the economic implications of these results?

Since $\bar{C}'(x) < 0$ for all $x \neq 0$, the rate of change of the average cost function is < 0 for all $x > 0$. That is, $\bar{C}(x)$ decreases as x increases.

Note that $\bar{C}(x)$ always lies above horizontal line $y = 20$, but gets arbitrarily close as $x \to \infty$:



Why? Because as the level of production increases, the fixed cost per unit of production drops. Then, the average cost approaches \$20 as $x \to \infty$.

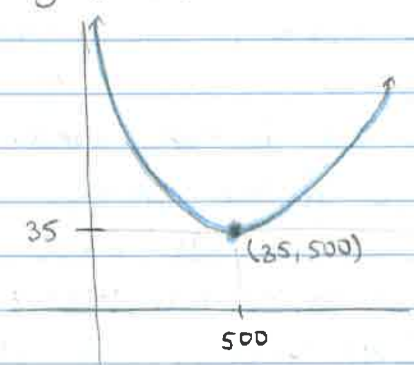
Note: Recall the cost function for making gluten-free cookies at Bob's Bakery:

$$C(x) = 0.0001x^3 - 0.08x^2 + 40x + 500$$

In this case, the average cost function is

$$\bar{C}(x) = \frac{C(x)}{x} = 0.0001x^2 - 0.08x + \frac{500}{x}$$

The graph of $\bar{C}(x)$ then looks like



The average cost of production decreases until $x = 500$, then increases.

Revenue Functions

Def. Revenue function $R(x)$ = revenue realized by a company from the sale of x units of a product.

If the company charges p dollars per unit, then
$R(x) = px$.

Note: Competitive market ⇒ price is p determined by market equilibrium

Monopoly ⇒ price is manipulated by controlling supply.

* Unit selling price p is related to quantity of the product demanded, x.
  ⇒ This relationship is given by the demand equation (discussed last week).

* Solving demand equation for p in terms of x yields unit price function f:
  $p = f(x)$

⇒ revenue function $R(x) = px = f(x)\,x$

Def. Marginal revenue = revenue realized from sale of an additional unit of the product, given that sales are already at a certain level.

Marginal revenue function = $R'(x)$.

Ex: Suppose relationship b/t unit price p and quantity demanded x is given by

$$p(x) = -0.02x + 400 \quad (0 \le x \le 20{,}000)$$

(a) Find the revenue function.

$$R(x) = px$$
$$= (-0.02x + 400)x$$
$$= -0.02x^2 + 400x$$

(b) Find the marginal revenue function

$$R'(x) = -0.04x + 400$$

(c) Compute $R'(2000)$ and interpret your result.

$$R'(2000) = -0.04(2000) + 400$$
$$= -80 + 400$$
$$= 320$$

The revenue realized from the sale of the 2001st unit is approx. \$320.

Profit Function

Def. Profit function P is given by

$$P(x) = R(x) - C(x), \quad x = \#\text{ units of product produced and sold.}$$

revenue function   cost function

Def. Marginal profit function $= P'(x)$

Approximates the profit or loss realized from sale of $(x+1)$st unit of product (assuming x units have been sold).

Ex: Suppose cost of producing x units of a product is

$$C(x) = 100x + 200,000,$$

and the relationship b/t unit price p and quantity demanded is

$$p(x) = -0.02x + 400 \quad (0 \le x \le 20,000)$$

(a) Find the profit function.

$$P(x) = R(x) - C(x)$$
$$= p(x)\,x - C(x)$$
$$= -0.02x^2 + 400x - (100x + 20,000)$$
$$= -0.02x^2 + 300x - 20,000$$

(b) Find the marginal profit function.

$$P'(x) = -0.04x + 300$$

(c) Compute $P'(2000)$ and interpet your result.

$$P'(2000) = -0.04(2000) + 300$$
$$= -80 + 300$$
$$= 220$$

$\Rightarrow$ profit realized from sale of 2001st unit is approx \$220.

(d) Graph the profit function.