6.2 Integration by Substitution

Rule: $\int f'(g(x)) \cdot g'(x)\, dx = f(g(x)) + C$

- We will "undo" the chain rule via u-substitution.

Ex: Consider $\int 2(2x+4)^5\, dx$.
How do we evaluate this integral?

Method 1 - Expand and integrate $\rightsquigarrow$ too long!

Method 2 - Use u-substitution to rewrite the integral in a simpler form:

Let $u = 2x+4$

$\Rightarrow du = 2\,dx$

$$\Rightarrow \int 2(2x+4)^5\, dx = \int \underbrace{(2x+4)^5}_{u^5} \cdot \underbrace{2\,dx}_{du}$$

$$= \int u^5\, du$$

$$= \frac{u^6}{6} + C$$

$$= \frac{(2x+4)^6}{6} + C$$

Check: $\frac{d}{dx}\left(\frac{(2x+4)^6}{6} + C\right) = (2x+4)^5 \cdot 2$ ✓

Integration by Substitution

1) let $u = g(x)$, where $g(x)$ is usually the inside of a composite function $f(g(x))$.
2) Find $du = g'(x)\,dx$
3) Use substitution to rewrite the integral entirely in terms of u.
4) Evaluate the resulting integral.
5) Replace u by $g(x)$ to obtain the final solution in terms of x.

Ex: Evaluate the integrals.

1) $\int 2x(x^2+3)^4\,dx$ $\qquad u = x^2+3$

$du = 2x\,dx$

$= \int u^4\,du = \frac{u^5}{5} + C$

$= \frac{(x^2+3)^5}{5} + C$

2) $\int 3\sqrt{3x+1}\,dx$ $\qquad u = 3x+1$

$du = 3\,dx$

$= \int \sqrt{u} = \int u^{1/2}$

$= \frac{u^{3/2}}{3/2} + C$

$= \frac{2}{3}u^{3/2} + C = \frac{2}{3}(3x+1)^{3/2} + C$

3) $\int x^2 (x^3+1)^{3/2}\, dx$

$u = x^3+1$

$du = 3x^2\, dx$

$\Rightarrow \frac{1}{3} du = x^2\, dx$

$= \frac{1}{3}\int u^{3/2}\, du$

$= \frac{1}{3}\cdot \frac{u^{5/2}}{5/2} + C = \frac{2}{15} u^{5/2} + C$

$= \frac{2}{15}(x^3+1)^{5/2} + C$

4) $\int e^{-3x}\, dx$

$u = -3x$

$du = -3\, dx$

$\Rightarrow -\frac{1}{3} du = dx$

$= -\frac{1}{3}\int e^u\, du$

$= -\frac{1}{3} e^u + C = -\frac{1}{3} e^{-3x} + C$

5) $\int \frac{x}{3x^2+1}\, dx$

$u = 3x^2+1$

$du = 6x\, dx$

$\Rightarrow \frac{1}{6} du = x\, dx$

$= \frac{1}{6}\int \frac{1}{u}\, du = \frac{1}{6}\ln|u| + C$

$= \frac{1}{6}\ln\underbrace{(3x^2+1)}_{\text{always } +} + C$

6) $\int \frac{(\ln x)^2}{2x}\, dx$

$u = \ln x$

$du = \frac{1}{x}\, dx$

$= \int \frac{u^2}{2}\, du$

$= \frac{1}{2}\cdot\frac{u^3}{3} + C = \frac{1}{6}(\ln x)^3 + C$

7) $\int \frac{3x^2+2}{x^3+2x+1}\,dx$ $\qquad u = x^3+2x+1$

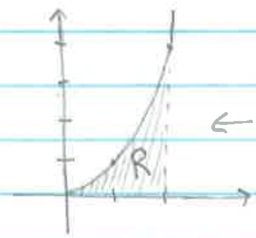
$du = (3x^2+2)\,dx$

$= \int \frac{1}{u}\,du = \ln|u| + C$

$= \ln|x^3+2x+1| + C$

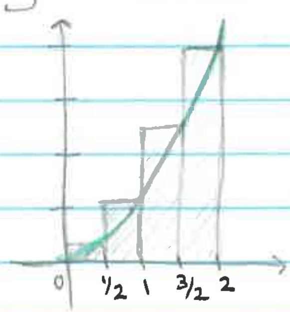
6.3 Area and the Definite Integral

Consider the function $f(x) = x^2$ on $[0, 2]$.
How do we find the area of the region under the graph of f for the given interval?



← want area of shaded region

Method: We can approximate the area by dividing the interval $[0, 2]$ into 4 subintervals and constructing 4 nonoverlapping rectangles.



We then approximate the area of R by adding the areas of the 4 rectangles:

$$A_R \approx \frac{1}{2} \cdot f(1/2) + \frac{1}{2} \cdot f(1) + \frac{1}{2} \cdot f(3/2) + \frac{1}{2} f(2)$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\right)^2 + (1)^2 + \left(\frac{3}{2}\right)^2 + (2)^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + 1 + \frac{9}{4} + 4\right]$$

$$= \frac{1}{2}\left(\frac{30}{4}\right) = \frac{30}{8}$$

Problem: Too large.

We could get closer to the area of R by dividing [0,2] into 8 intervals:

$\Rightarrow$ The more intervals we divide [0,2] into, the closer we get to the area of R.

Area Under the Graph of a Function

Let f be a cts (nonneg) function on [a,b]. Then the area of the region under the graph of f is

$$A = \lim_{n\to\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x, \quad \text{where}$$

$x_1, x_2, \ldots, x_n$ are points on the n subintervals of [a,b],

Ex:

$\Delta x = \dfrac{b-a}{n}$ = width of each subinterval

The Definite Integral

The definite integral of f from a to b is:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} [f(x_1) + f(x_2) + \cdots + f(x_n)]\Delta x, \quad \text{where}$$

a = lower limit of integration

b = upper limit of integration

Note: f is integrable on [a,b] ($\int_a^b f(x)\,dx$ exists) if f is continuous on [a,b].