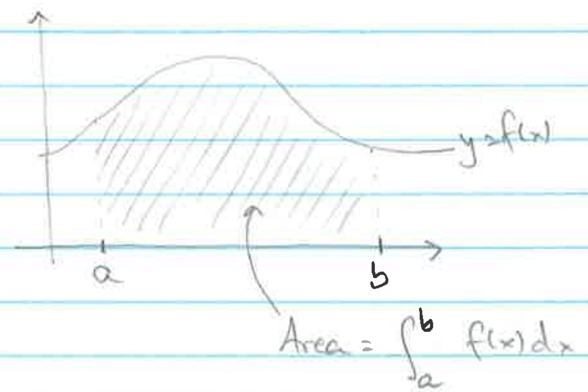
Geometric Interpretation of the Definite Integral

If f is nonneg and cts on $[a,b]$, then

$$\int_a^b f(x)\,dx = \text{area of the region under the graph of } f \text{ on } [a,b]$$



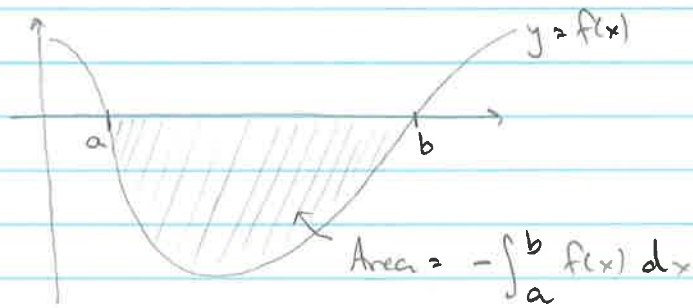
Remark: If $\int_a^b f(x)\,dx$ is defined and $a \le c \le b$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

HW: Explain why (using pictures if necessary)

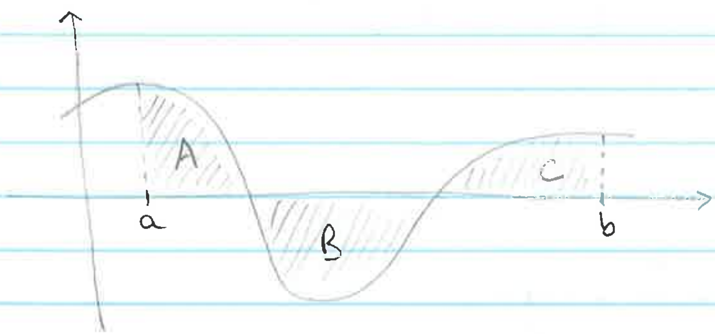
Prop: If f is negative and cts on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \left[f(x_1) + f(x_2) + \cdots + f(x_n) \right] \Delta x < 0$$



HW: Prove this

Prop: If f is cts on $[a,b]$, then $\int_a^b f(x)\,dx$ is equal to the area of the region above $[a,b]$ minus the area of the region below $[a,b]$:



$$\Rightarrow \int_a^b f(x)\,dx = A - B + C$$

6.4 The Fundamental Theorem of Calculus

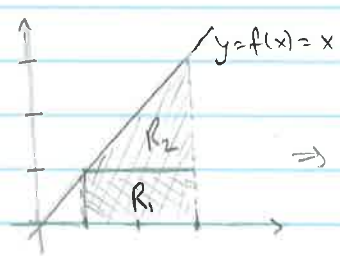
Theorem (TFToC) - Let f be a cts function on $[a,b]$. Then, $\int_a^b f(x)\,dx = F(b) - F(a)$,

where $F(x)$ is any antiderivative of $f(x)$ $(F'(x) = f(x))$.

Notation: $\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$

Ex: 1) Let R be the region under the graph of $f(x) = x$ on the interval $[1,3]$. Use the FToC to find the area of R, and verify your result by elementary means:

$$A_R = \int_1^3 x\,dx = \frac{x^2}{2} + C\Big|_1^3 = \left(\frac{3^2}{2} + C\right) - \left(\frac{1}{2} + C\right) = \frac{3^2 - 1}{2} = 4$$



$$\Rightarrow A_R = A_{R_1} + A_{R_2}$$
$$= (1)(2) + \frac{(2)(2)}{2} = 2 + 2 = 4 \checkmark$$

2) Find the area of the region R under the graph of $y = x^2 + 1$ from $x = -1$ to $x = 2$.

$$A_R = \int_{-1}^{2} x^2 + 1 = \frac{x^3}{3} + x\Big|_{-1}^{2}$$
$$= \left(\frac{2^3}{3} + 2\right) - \left(\frac{-1}{3} - 1\right)$$
$$= \frac{8}{3} + 2 + \frac{1}{3} + 1 = 6$$

Evaluating Definite Integrals

Ex: 1) $\int_1^3 (3x^2 + e^x)\,dx = x^3 + e^x \Big|_1^3$

$= (3^3 + e^3) - (1 + e)$

$= 27 + e^3 - 1 - e$

$= 26 + e^3 - e$

2) $\int_1^2 \left(\frac{1}{x} - \frac{1}{x^2}\right) dx = \ln|x| + \frac{1}{x}\Big|_1^2$

$= (\ln|2| + 1/2) - (\cancel{\ln|1|} + 1)$

$= \ln|2| - 1/2$

6.5 Evaluating Definite Integrals

Properties of the Definite Integral

Let f, g be integrable functions, then

1) $\int_a^a f(x)\,dx = 0$

2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

4) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)$

5) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad (a<c<b)$

Method of Substitution for Definite Integrals

Ex 1: Evaluate $\int_0^4 x\sqrt{9+x^2}\,dx$.

Method 1

$u = 9+x^2, \quad du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$

$$\Rightarrow \int x\sqrt{9+x^2}\,dx = \frac{1}{2}\int \sqrt{u}\,du$$

$$= \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C = \frac{1}{3}u^{3/2} + C$$

$$\Rightarrow \int x\sqrt{9+x^2}\,dx = \frac{1}{3}(9+x^2)^{3/2} + C$$

$$\Rightarrow \int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{3}(9+x^2)^{3/2}\Big|_0^4$$

$$= \frac{1}{3}\left[(9+16)^{3/2} - 9^{3/2}\right]$$

$$= \frac{1}{3}(125-27)$$

$$= \frac{98}{3}$$

Method 2: Changing the limits of integration

$u = 9 + x^2$, $du = 2x\,dx \Rightarrow \frac{1}{2}du = x\,dx$

$\Rightarrow \int x\sqrt{9+x^2}\,dx = \frac{1}{2}\int \sqrt{u}\,du$

$\Rightarrow \int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{2}\int_{u_1 = 9+0^2}^{u_2 = 9+4^2} \sqrt{u}\,du$

changed limits of integration

$= \frac{1}{2}\int_9^{25} u^{1/2}\,du$

$= \frac{1}{3} u^{3/2}\Big|_9^{25}$

$= \frac{1}{3}\left(25^{3/2} - 9^{3/2}\right)$

$= \frac{98}{3}$

⚠ When using Method 2, you must make sure to adjust the limits of integration to reflect integrating wrt the new variable u.

Ex: 1) Evaluate $\int_0^2 xe^{2x^2}\,dx$ $u = 2x^2$

$du = 4x\,dx \Rightarrow \frac{1}{4}du = x\,dx$

$= \frac{1}{4}\int_{2(0)^2}^{2(2)^2} e^u\,du$

$= \frac{1}{4}e^u\Big|_0^8 = \frac{1}{4}\left(e^8 - e^0\right) = \frac{1}{4}\left(e^8 - 1\right)$

2) Evaluate $\int_0^1 \frac{x^2}{x^3+1}\,dx$ $u = x^3 + 1$

$du = 3x^2\,dx \Rightarrow \frac{1}{3}du = x^2\,dx$

$= \frac{1}{3}\int_{0^3+1}^{1^3+1} \frac{1}{u}\,du$

$= \frac{1}{3}\ln|u|\Big|_1^2 = \frac{1}{3}\left(\ln(2) - \cancel{\ln(1)}^{\,0}\right) = \frac{1}{3}\ln(2)$