6.5 Evaluating Definite Integrals

Properties of the Definite Integral

Let f, g be integrable functions, then

1) $\int_a^a f(x)\,dx = 0$

2) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

3) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

4) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)$

5) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad (a < c < b)$

Method of Substitution for Definite Integrals

Ex 1: Evaluate $\int_0^4 x\sqrt{9+x^2}\,dx$.

Method 1

$u = 9+x^2, \quad du = 2x\,dx \Rightarrow \frac{1}{2}\,du = x\,dx$

$$\Rightarrow \int x\sqrt{9+x^2}\,dx = \frac{1}{2}\int \sqrt{u}\,du$$

$$= \frac{1}{2}\frac{u^{3/2}}{3/2} + C = \frac{1}{3}u^{3/2} + C$$

$$\Rightarrow \int x\sqrt{9+x^2}\,dx = \frac{1}{3}(9+x^2)^{3/2} + C$$

$$\Rightarrow \int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{3}(9+x^2)^{3/2}\Big|_0^4$$

$$= \frac{1}{3}\left[(9+16)^{3/2} - 9^{3/2}\right]$$

$$= \frac{1}{3}(125-27)$$

$$= \frac{98}{3}$$

Method 2: Changing the limits of integration

$$u = 9 + x^2 \quad , \quad du = 2x\,dx \quad \Rightarrow \quad \tfrac{1}{2}\,du = x\,dx$$

$$\Rightarrow \int x\sqrt{9+x^2}\,dx = \frac{1}{2}\int \sqrt{u}\,du$$

$$\Rightarrow \int_0^4 x\sqrt{9+x^2}\,dx = \frac{1}{2}\int_{u_1 = 9+0^2}^{u_2 = 9+4^2} \sqrt{u}\,du$$

changed limits of integration

$$= \frac{1}{2}\int_9^{25} u^{1/2}\,du$$

$$= \frac{1}{3}\,u^{3/2}\Big|_9^{25}$$

$$= \tfrac{1}{3}\left(25^{3/2} - 9^{3/2}\right)$$

$$= \frac{98}{3}$$

⚠ When using Method 2, you must make sure to adjust the limits of integration to reflect integrating wrt the new variable u.

Ex: 1) Evaluate $\displaystyle\int_0^2 xe^{2x^2}\,dx$ $\qquad u = 2x^2$

$du = 4x\,dx \Rightarrow \frac{1}{4}\,du = x\,dx$

$$= \frac{1}{4}\int_{2(0)^2}^{2(2)^2} e^u\,du$$

$$= \frac{1}{4}e^u\Big|_0^8 = \frac{1}{4}\left(e^8 - e^0\right) = \frac{1}{4}\left(e^8 - 1\right)$$

2) Evaluate $\displaystyle\int_0^1 \frac{x^2}{x^3+1}\,dx$ $\qquad u = x^3 + 1$

$du = 3x^2\,dx \Rightarrow \frac{1}{3}\,du = x^2\,dx$

$$= \frac{1}{3}\int_{0^3+1}^{1^3+1} \frac{1}{u}\,du$$

$$= \frac{1}{3}\ln|u|\Big|_1^2 = \frac{1}{3}\left(\ln(2) - \cancel{\ln(1)}^{\,0}\right) = \frac{1}{3}\ln(2)$$

Finding the Area Under a Curve

Ex: Find the area of the region R under the graph of $f(x) = e^{\frac{1}{2}x}$ from $x=-1$ to $x=1$.

$$A_R = \int_{-1}^{1} e^{\frac{1}{2}x}\,dx \qquad u = \tfrac{1}{2}x, \quad du = \tfrac{1}{2}\,dx$$

$$\Rightarrow 2\,du = dx$$

$$= 2\int_{\frac{1}{2}(-1)}^{\frac{1}{2}(1)} e^{u}\,du$$

$$= 2e^{u}\Big|_{-1/2}^{1/2} = 2\left(e^{1/2} - e^{-1/2}\right)$$

HW: Let f be the function defined by

$$f(x) = \begin{cases} \sqrt{x}, & 0 \le x \le 1 \\ x^2, & 1 \le x \le 2. \end{cases}$$

Find the area of the region R under the graph of $f(x)$ from $x=0$ to $x=2$.

Average Value of a Function

Recall: The average value of a set of n numbers $y_1, y_2, \ldots, y_n$ is

$$\frac{y_1 + y_2 + \cdots + y_n}{n}.$$

Suppose f is a cts function on $[a,b]$. We can divide $[a,b]$ into n subintervals of equal length, $(b-a)/n$.

Choose point x_i for each i^{th} subinterval.

Then the average value of numbers $f(x_1), f(x_2), \ldots, f(x_n)$ is given by

$$\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}$$

rewriting

$$= \frac{b-a}{b-a}\left[f(x_1)\cdot\frac{1}{n} + f(x_2)\cdot\frac{1}{n} + \cdots + f(x_n)\cdot\frac{1}{n}\right]$$

$$= \frac{1}{b-a}\left[f(x_1)\cdot\frac{b-a}{n} + f(x_2)\cdot\frac{b-a}{n} + \cdots + f(x_n)\cdot\frac{b-a}{n}\right]$$

$$= \frac{1}{b-a}\left[f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x\right]$$

- As n gets larger, this expression approximates the average value of $f(x)$ over $[a,b]$ w/ increasing accuracy.

$\Rightarrow$ average value of $f(x)$ over $[a,b]$ $= \lim_{n\to\infty}\left[\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}\right]$

$$= \frac{1}{b-a}\lim_{n\to\infty}\left[f(x_1)\,\Delta x + f(x_2)\,\Delta x + \cdots + f(x_n)\,\Delta x\right]$$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

Def. If f is integrable on $[a,b]$, the <u>average value</u> of f over $[a,b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,dx$$

Ex. Find the average value of $f(x)=\sqrt{x}$ over the interval $[0,4]$.

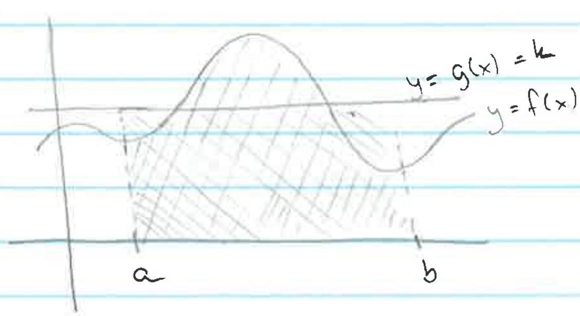
$$\frac{1}{4-0}\int_0^4 \sqrt{x}\,dx = \frac{1}{4}\int_0^4 x^{1/2}\,dx$$

$$= \frac{1}{4}\,\frac{x^{3/2}}{3/2}\Big|_0^4$$

$$= \frac{1}{6}\,x^{3/2}\Big|_0^4 = \frac{1}{6}\cdot 4^{3/2} = \frac{4}{3}$$

Geometric interpretation of average value of f over $[a,b]$:

- If f is nonneg., $\int_a^b f(x)\,dx$ gives the area under the graph of f.
- We can replace $f(x)$ by constant function $g(x)=k$ s.t. the areas under f and g over $[a,b]$ are the same:



→ Since the area under g over $[a,b]$ is $k(b-a)$,

$$k(b-a) = \int_a^b f(x)\,dx$$

$$\Rightarrow \quad k = \frac{1}{b-a}\int_a^b f(x)\,dx$$

- Thus, the average value of f over $[a,b]$ is the height of the rectangle w/ base length $b-a$ that has the same area as that of the region under f from $x = a$ to $x = b$.