Elasticity of Demand

Elasticity of demand measures how a small percentage change in the price of a product affects the percentage change in the quantity demanded of that product.

Def. $\underline{\text{Relative change (of the size of a quantity)}} = \dfrac{\text{change in the size of the quantity}}{\text{size of the quantity}}$

Ex: Suppose that the death rate of GoT characters increases from the current rate of 50% per season to 55% per season.

$\Rightarrow$ the relative change in death rate is

$$\frac{5}{50} = \frac{1}{10} = 0.1 \Rightarrow 10\%$$

But if current death rate is 40% per season, then a change of 5% per season to 45% per season would yield relative change

$$\frac{5}{40} = 0.125 \Rightarrow 12.5\%$$

Def. $\underline{\text{Relative rate of change (of a quantity)}} = \dfrac{\text{rate of change of size of quantity}}{\text{size of quantity}}$

Ex: Suppose that the current income of a mime of \$20,000 per year is changing at the rate of 12% per year.

$\Rightarrow$ relative rate of change of income is

$$\frac{12}{20{,}000} = 0.0006 \Rightarrow 0.06\%$$

Recall: $f'(x)$ = rate of change of $f(x)$ at x

$\Rightarrow$ relative rate of change of $f(x)$ at x $= \dfrac{f'(x)}{f(x)}$

or $100\dfrac{f'(x)}{f(x)}\ \%$

Rewriting demand function: $x = f(p)$

x: quantity demanded; p: unit price

* Since the quantity demanded of a product typically decreases as unit price increases, f is a decreasing function of p.

$$\frac{\%\text{ rate of change of } f \text{ at } p}{\%\text{ rate of change of } p} = \frac{\dfrac{100 f'(p)}{f(p)}}{\dfrac{100\frac{d}{dp}(p)}{p}}$$

$$= \frac{f'(p)}{f(p)} \cdot \frac{p}{\underbrace{\frac{d}{dp}(p)}_{\to 1}}$$

$$= \frac{p f'(p)}{f(p)}$$

* Economists call the negative of this quantity the elasticity of demand.

Def. If f is a (differentiable) demand function defined by $x = f(p)$, the <u>elasticity of demand</u> at price p is given by

$$E(p) = -\frac{p f'(p)}{f(p)}$$

Ex: Consider demand equation
$p = -0.02x + 400 \quad (0 \le x \le 20{,}000)$.

(a) Find elasticity of demand, $E(p)$.

Solve for x:

$$p = -0.02x + 400 \xRightarrow{-400} p - 400 = -0.02x \quad \left(= -\tfrac{2}{100}\right)$$

$$\xRightarrow{\div(-0.02)} -50p + 20{,}000 = x$$

$$\Rightarrow x = f(p) = -50p + 20{,}000$$

$$E(p) = -\frac{p\,f'(p)}{f(p)}$$

$$= -\frac{p(-50)}{-50p + 20{,}000}$$

$$= \frac{50p}{50(400-p)}$$

$$= \frac{p}{400-p}$$

(b) Compute $E(100)$, and interpret your result.

$$E(100) = -\frac{100}{400-100} = -\frac{1}{3}$$

$$\Rightarrow E(100) = -\frac{\%\text{ change in quantity demanded when } p=100}{\%\text{ change in unit price when } p=100}$$

$\Rightarrow$ When the unit price is \$100 dollars, an increase of 1% in the unit price will cause a decrease of $\approx 0.33\%$ in the quantity demanded.

(c) Compute $E(300)$ and interpret your result.

$$E(300) = -\frac{300}{400-300} = -3$$

$\Rightarrow$ When the unit price is \$300 dollars, an increase of 1% in the unit price will cause a decrease of 3% in the quantity demanded.

Def. The demand is said to be <u>elastic</u> if $|E(p)| > 1$.

<u>unitary</u> if $|E(p)| = 1$.

<u>inelastic</u> if $|E(p)| < 1$.

Ex: In previous example, demand was elastic when $p = 300$, but inelastic when $p = 100$.

3.6 Implicit Differentiation and Related Rates

Implicit equation example: $(x^2+1)y = x^2-1$

Explicit equation: $y = f(x) = \dfrac{x^2-1}{x^2+1}$

Q: How does one find the derivative of an implicit equation without solving for y?

A: We use implicit differentiation.

Ex: 1) Given $y^2 = x$, find $\dfrac{dy}{dx}$.

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x)$$

Recall that y is a function of x, that is, $y = f(x)$, so we can write

$$\frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2$$

$$= 2f(x)\cdot f'(x), \text{ by Chain Rule}$$

$$= 2y\cdot\frac{dy}{dx}, \text{ since } f'(x) = \frac{d}{dx}f(x) = \frac{dy}{dx}$$

$$\Rightarrow 2y\frac{dy}{dx} = \frac{d}{dx}(x) = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}, \text{ solving for } \frac{dy}{dx}$$

Finding $\dfrac{dy}{dx}$ by ID:

1) Differentiate both sides with respect to x
2) Solve the resulting equation for $\dfrac{dy}{dx}$ in terms of x and y.

Ex: 2) Find $\frac{dy}{dx}$ given $y^3 - y + 2x^3 - x = 8$

$$\frac{d}{dx}\left(y^3 - y + 2x^3 - x\right) = \frac{d}{dx}(8)$$

$$\Rightarrow \frac{d}{dx}\left([f(x)]^3 - f(x) + 2x^3 - x\right) = \frac{d}{dx}(8), \quad \text{since } y = f(x)$$

$$\Rightarrow 3[f(x)]^2 f'(x) - f'(x) + 6x^2 - 1 = 0, \quad \text{by chain rule}$$

$$\Rightarrow 3y^2 \frac{dy}{dx} - \frac{dy}{dx} + 6x^2 - 1 = 0, \quad \text{since } f(x) = y \text{ and } f'(x) = \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx}\left(3y^2 - 1\right) = 1 - 6x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1 - 6x^2}{3y^2 - 1}$$

3) Consider $x^2 + y^2 = 4$

a. Find dy/dx by ID.

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(4)$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y} \quad (y \neq 0)$$

b. Find the slope of the tangent line to the graph of function $y = f(x)$ at $(1, \sqrt{3})$

$$m = \left.\frac{dy}{dx}\right|_{(1,\sqrt{3})} = \left.\frac{-x}{y}\right|_{(1,\sqrt{3})} = \frac{-1}{\sqrt{3}}$$

c. What is the equation of the tangent line to $y = f(x)$ at $(1, \sqrt{3})$?

$$y - y_0 = m(x - x_0)$$

$$\Rightarrow y - \sqrt{3} = \frac{-1}{\sqrt{3}}(x - 1)$$

Ex: 4) Find $\frac{dy}{dx}$ at $(1, 2)$ given $x^2y^3 + 6x^2 = y + 12$

$$\frac{d}{dx}\left(x^2y^3 + 6x^2\right) = \frac{d}{dx}\left(y + 12\right)$$

$$\Rightarrow \frac{d}{dx}(x^2y^3) + \frac{d}{dx}(6x^2) = \frac{d}{dx}(y) + \frac{d}{dx}(12)$$

$$\Rightarrow \left[\frac{d}{dx}(x^2)y^3 + x^2\frac{d}{dx}(y^3)\right] + 12x = \frac{dy}{dx}$$

$$\Rightarrow \left[2xy^3 + x^2 \cdot 3y^2\frac{dy}{dx}\right] + 12x = \frac{dy}{dx}$$

$$\Rightarrow \left(2xy^3 + 3x^2y^2\frac{dy}{dx} + 12x\right)\bigg|_{(1,2)} = \frac{dy}{dx}\bigg|_{(1,2)}$$

$$\Rightarrow 2(1)(2)^3 + 3(1)^2(2)^2\frac{dy}{dx}\bigg|_{(1,2)} + 12(1) = \frac{dy}{dx}\bigg|_{(1,2)}$$

$$\Rightarrow 16 + 12\frac{dy}{dx}\bigg|_{(1,2)} + 12 = \frac{dy}{dx}\bigg|_{(1,2)}$$

$$\Rightarrow 11\frac{dy}{dx}\bigg|_{(1,2)} = -28$$

$$\Rightarrow \frac{dy}{dx}\bigg|_{(1,2)} = \frac{-28}{11}$$

* see 91.5

Related Rates

Ex: The number of housing starts in the southwest, $N(t)$ (in units of a million), over the next 5 years is related to the mortgage rate $r(t)$ (% per year) by the equation

$$9N^2 + r = 36.$$

What is the rate of change in the # of housing starts wrt time when mortgage rate is 11% per year and is increasing at the rate of 1.5% per year?

Given: $r(t) = 11$, $\frac{dr}{dt} = 1.5$

Want: $\frac{dN}{dt}$

$$9N^2 + r = 36$$

$$\Rightarrow \frac{d}{dt}\left(9N^2 + r\right) = \frac{d}{dt}(36)$$

$$\Rightarrow \frac{d}{dt}\left(9N^2\right) + \frac{d}{dt}(r) = 0$$

$$\Rightarrow 18N\frac{dN}{dt} + \frac{dr}{dt} = 0$$

Need: $N(t)$

$$9N^2 + r = 36$$

$$\Rightarrow 9N^2 + 11 = 36$$

$$\Rightarrow N^2 = \frac{25}{9} \Rightarrow N = \frac{5}{3}$$

$$18\left(\frac{5}{3}\right)\frac{dN}{dt} + 1.5 = 0$$

$$\Rightarrow 30\frac{dN}{dt} = -1.5$$

$$\Rightarrow \frac{dN}{dt} = \frac{-1.5}{30} = -0.05$$