c. What is the equation of the tangent line to $y = f(x)$ at $(1, \sqrt{3})$?

$$y - y_0 = m(x - x_0)$$

$$\Rightarrow y - \sqrt{3} = \frac{-1}{\sqrt{3}}(x - 1)$$

Ex: 4) Find $\frac{dy}{dx}$ at $(1,2)$ given $x^2y^3 + 6x^2 = y + 12$

$$\frac{d}{dx}\left(x^2y^3 + 6x^2\right) = \frac{d}{dx}\left(y + 12\right)$$

$$\Rightarrow \frac{d}{dx}(x^2y^3) + \frac{d}{dx}(6x^2) = \frac{d}{dx}(y) + \frac{d}{dx}(12)$$

$$\Rightarrow \left[\frac{d}{dx}(x^2)\,y^3 + x^2\frac{d}{dx}(y^3)\right] + 12x = \frac{dy}{dx}$$

$$\Rightarrow \left[2xy^3 + x^2 \cdot 3y^2\frac{dy}{dx}\right] + 12x = \frac{dy}{dx}$$

$$\Rightarrow \left(2xy^3 + 3x^2y^2\frac{dy}{dx} + 12x\right)\Big|_{(1,2)} = \frac{dy}{dx}\Big|_{(1,2)}$$

$$\Rightarrow 2(1)(2)^3 + 3(1)^2(2)^2\frac{dy}{dx}\Big|_{(1,2)} + 12(1) = \frac{dy}{dx}\Big|_{(1,2)}$$

$$\Rightarrow 16 + 12\frac{dy}{dx}\Big|_{(1,2)} + 12 = \frac{dy}{dx}\Big|_{(1,2)}$$

$$\Rightarrow 11\frac{dy}{dx}\Big|_{(1,2)} = -28$$

$$\Rightarrow \frac{dy}{dx}\Big|_{(1,2)} = \frac{-28}{11}$$

* see 91.5

Related Rates

Ex: The number of housing starts in the southwest, $N(t)$ (in units of a million), over the next 5 years is related to the mortgage rate $r(t)$ (% per year) by the equation

$$9N^2 + r = 36.$$

What is the rate of change in the # of housing starts wrt time when mortgage rate is 11% per year and is increasing at the rate of 1.5% per year?

Given: $r(t) = 11$, $\frac{dr}{dt} = 1.5$

Want: $\frac{dN}{dt}$

$$9N^2 + r = 36$$

$$\Rightarrow \frac{d}{dt}(9N^2 + r) = \frac{d}{dt}(36)$$

$$\Rightarrow \frac{d}{dt}(9N^2) + \frac{d}{dt}(r) = 0$$

$$\Rightarrow 18N\frac{dN}{dt} + \frac{dr}{dt} = 0$$

Need: $N(t)$

$$9N^2 + r = 36$$

$$\Rightarrow 9N^2 + 11 = 36$$

$$\Rightarrow N^2 = \frac{25}{9} \quad \Rightarrow N = \frac{5}{3}$$

$$18\left(\frac{5}{3}\right)\frac{dN}{dt} + 1.5 = 0$$

$$\Rightarrow 30\frac{dN}{dt} = -1.5$$

$$\Rightarrow \frac{dN}{dt} = \frac{-1.5}{30} = -0.05$$

Implicit differentiation for 2nd Derivative.

Example: Find $\frac{d^2y}{dx^2}$ when

$$2x^2 + y = xy.$$

1st deriv: $2x\frac{dx}{dx} + \frac{dy}{dx} = \frac{dx}{dx}y + x\frac{dy}{dx} = y + x\frac{dy}{dx}$

$$\Rightarrow \quad 2x + \frac{dy}{dx} = y + x\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}(1-x) = y - 2x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y-2x}{1-x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{\left(\frac{dy}{dx} - 2\right)(1-x) - (y-2x)(-1)}{(1-x)^2}$$

$$= \frac{\left[\left(\frac{y-2x}{1-x}\right) - 2\right](1-x) + y - 2x}{(1-x)^2}$$

Solving Related Rates Problems

1) Assign a variable to each quantity. Draw a diagram if needed.
2) Write the given values of the variables and their rates of change wrt t
3) Find an equation giving the relationship b/t the variables.
4) Differentiate both sides of this equation implicitly wrt t
5) Replace the variables and their derivatives by the numerical data found in Step 2, and solve the equation for the desired rate of change.

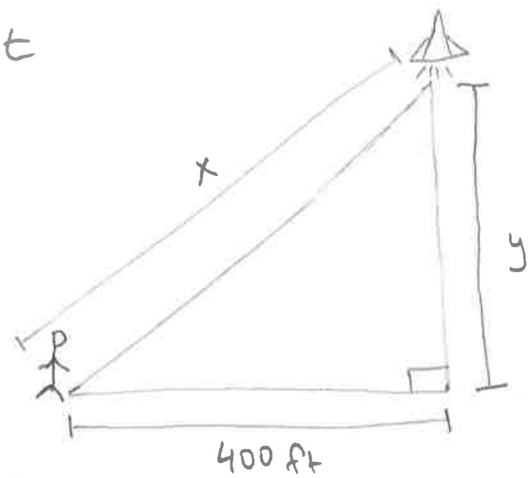
⚠ Do not replace the variables in the equation found in Step 3 by their numerical values before differentiating the equation.

Example 2: A distance of 4000 ft from the launch site, a spectator is observing a rocket being launched. If the rocket lifts off vertically and is rising at a speed of 600 ft/s when it is at an altitude of 3000 ft, how fast is the distance between the rocket and the spectator changing at that instant?

Let, $x =$ altitude of the rocket
$y =$ distance b/t rocket and spectator

at time t

functions of t



Given: $\frac{dy}{dt} = 600$ when $y = 3000$

Want: $\frac{dx}{dt}$

$$\Rightarrow 400^2 + y^2 = x^2$$

$$\Rightarrow \frac{d}{dt}\left(400^2 + y^2 = x^2\right)$$

$$\Rightarrow 2y\frac{dy}{dt} = 2x\frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$$

$\rightsquigarrow$ Need x!

$$x^2 = y^2 + 400^2 = 3000^2 + 400^2$$

$$\Rightarrow x = \sqrt{3000^2 + 400^2} = 5000$$

$$= \frac{3000}{5000} \cdot 600 = \frac{1800}{5} = \boxed{360.}$$