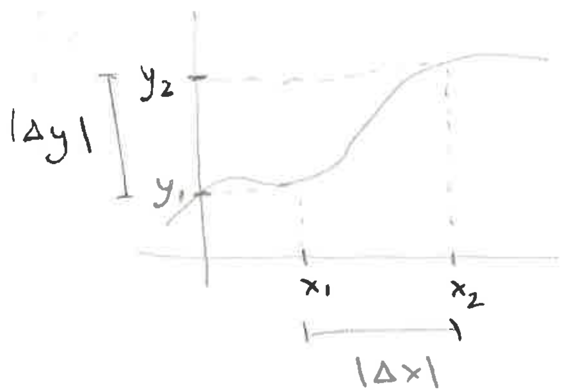
3.7 Differentials

Let x = variable quantity, and suppose x changes from x_1 to x_2. Then

$$\Delta x = \underbrace{x_2}_{\text{final value}} - \underbrace{x_1}_{\text{initial value}}$$

Ex:



$\Delta y = y_2 - y_1$

$\Delta x = x_2 - x_1$

Def. Let $y = f(x)$ be a differentiable function of x. Then,

1. The differential dx of independent variable x is
$$dx = \Delta x$$
2. " " dy " dependent variable y is
$$dy = \Delta y$$

Ex: a) Find the differential dy of the function
$$f(x) = x^2 + 3$$
$$\rightarrow \frac{dy}{dx} = f'(x) = 2x$$
$$\rightarrow dy = 2x\,dx$$

b) Use differentials to approximate Δy when x changes from 2 to 2.01

end

start

$\rightarrow \Delta x = 2.01 - 2 = 0.01$

$dy = 2x\,dx \approx 2x\,\Delta x = 2(2)(0.01) = 0.04$

input start value

... If x changes from 2 to 1.98?

$\Rightarrow \Delta x = 1.98 - 2 = -0.02$

$\Rightarrow dy \approx 2x\,\Delta x = 2(2)(-0.02) = -0.08$

c) Use differentials to approximate the quantity

$\sqrt{26.5}$

Let $y = f(x) = \sqrt{x}$, Note: 25 is nearest perfect square, so take $x_1 = 25$

$x_2 = 26.5$

$\Rightarrow \Delta x = 1.5$

$\Rightarrow \frac{dy}{dx} = \frac{1}{2} x^{-1/2}$

$\Rightarrow dy = \frac{1}{2} x^{-1/2}\,dx$

$\Rightarrow \Delta y \approx \frac{1}{2\sqrt{x}} \Delta x$

$= \frac{1}{2\sqrt{25}} (1.5)$

$= \frac{1}{10} (1.5)$

$= 0.15$

$\Rightarrow \sqrt{26.5} - \sqrt{25} = \Delta y \approx 0.15$

$\Rightarrow \sqrt{26.5} \approx 0.15 + \sqrt{25}$

Ex: A sphere of radius 3 in is painted w/ coat of paint 0.02 in. Approximate the amount of paint used (to nearest cubic inch).

$$V_{sphere} = \frac{4}{3}\pi r^3$$

Want ΔV, when $\Delta r = 0.02$

$$dV = 4\pi r^2 \, dr$$

$$\Rightarrow \Delta V \approx 4\pi (3)^2 \Delta x = 4\pi \cdot 9 \cdot (0.02)$$

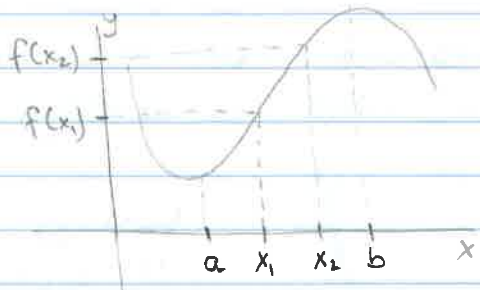
$$= 0.72\pi$$

4.1 Applications of the First Derivative

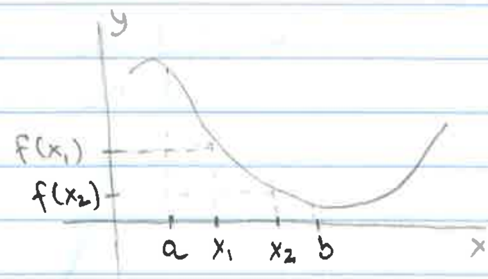
Increasing and Decreasing Functions

Def. A function f is increasing (resp. decreasing) on an interval (a,b) if for all x_1, x_2 in (a,b), $f(x_1) < f(x_2)$ (resp. $f(x_1) > f(x_2)$) whenever $x_1 < x_2$.

Ex:



increasing



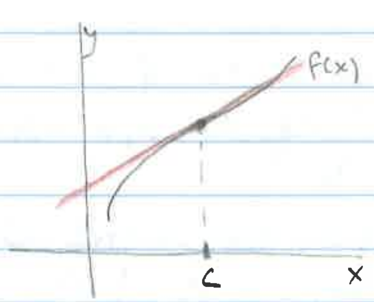
decreasing

Def. We say f is increasing (resp. decreasing) at a number c if there exists an interval (a,b) containing c s.t. f is increasing (resp. decreasing) on (a,b)

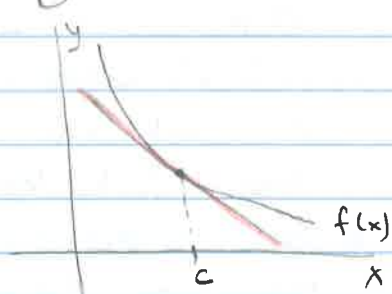
Fact - If the derivative at a point is positive, the slope of the tangent line at that point is positive, and the function is increasing

- Likewise for decreasing.

Ex:



increasing



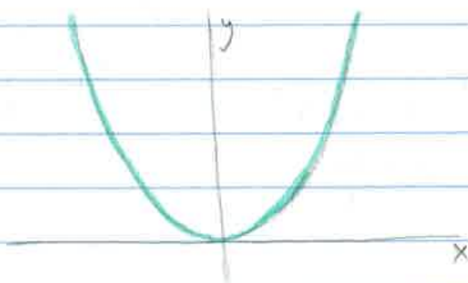
decreasing

Theorem

1) If $f'(x) > 0$ (resp. $f'(x) < 0$) for every value of x in an interval (a,b), then f is increasing (resp. decreasing) on (a,b).

2) If $f'(x) = 0$ for every value of x in an interval (a,b), then f is constant on (a,b).

Ex: Find the interval where $f(x) = x^2$ is increasing and the interval where it is decreasing



increasing: $(0, \infty)$
decreasing: $(-\infty, 0)$

$f'(x) = 2x \Rightarrow f'(x) > 0$ for $x > 0$
$f'(x) < 0$ for $x < 0$
$\Rightarrow$ increasing at $(0, \infty)$
decreasing at $(-\infty, 0)$

Note: A continuous function cannot change sign unless it equals zero for some value of x.

Determining Intervals Where a Function is Increasing or Decreasing

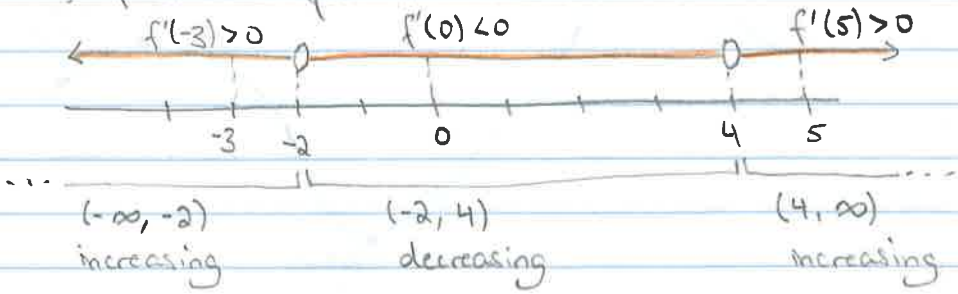
ex

1) Find all values of x for which $f'(x) = 0$ or f' is discontinuous, and identify the open intervals determined by these numbers.
2) Select a test number c in each interval found in step 1, and determine the sign of $f'(c)$ in that interval.
   a. If $f'(c) > 0$, f is increasing on that interval
   b. If $f'(c) < 0$, f is decreasing on that interval

Ex: Determine the intervals where
$f(x) = x^3 - 3x^2 - 24x + 32$
is increasing, and where it is decreasing.

$f'(x) = 3x^2 - 6x - 24$
$= 3(x^2 - 2x - 8)$
$= 3(x-4)(x+2)$

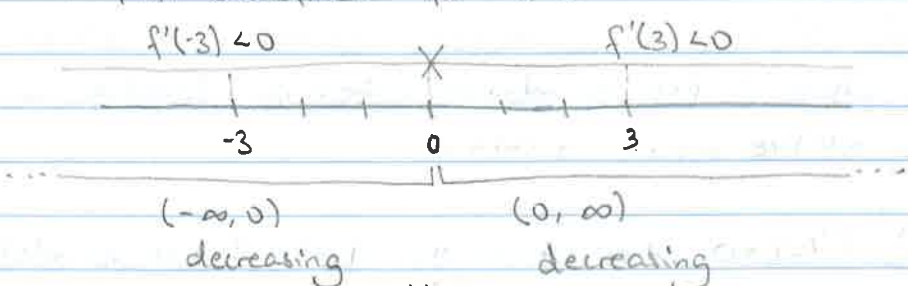
$\Rightarrow f'(x) = 0$ for $x = 4$ and $x = -2$

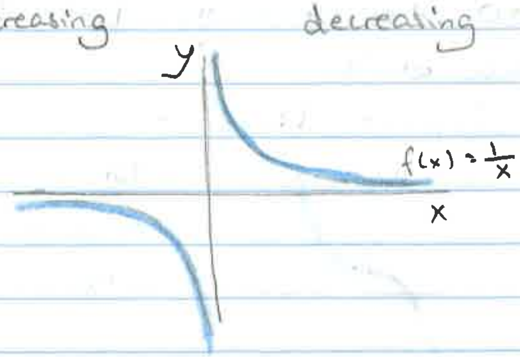


Ex: Determine the intervals where $f(x) = \frac{1}{x}$
is increasing and where it is decreasing.

$f'(x) = -x^{-2} = \frac{-1}{x^2}$

$\Rightarrow f'(x)$ undefined for $x = 0$

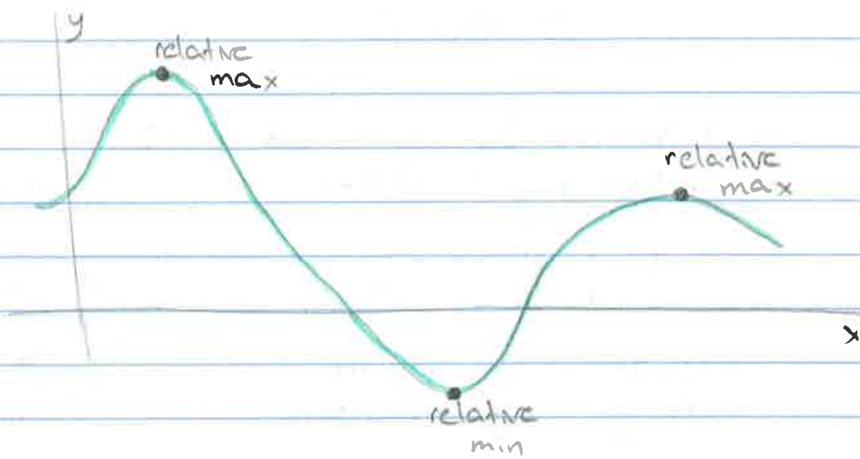




Relative Extrema

Def. The relative maxima and relative minima of a function correspond to the "high points" and "low points" of a function.

Ex:



Def. A function f has a relative maximum (resp. relative minimum) at $x=c$ if there exists an open interval (a,b) containing c s.t. $f(x) \le f(c)$ (resp. $f(x) \ge f(c)$) for all x in (a,b).

Note: If f is a differentiable function, then at any c where f has a relative extremum (relative min or relative max), $f'(c)=0$.

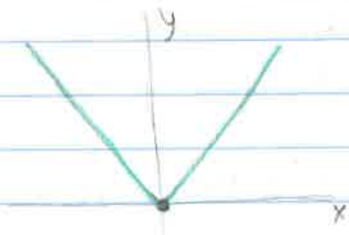
⚠ $f'(c)=0$ does not imply that f has a relative extrema at c

Ex: Consider $f(x)=x^3$

$f'(0)=0$ but f does not have a relative extremum at 0.

Note: A relative extremum of a function may exist at a point where the derivative does not exist.

Ex:

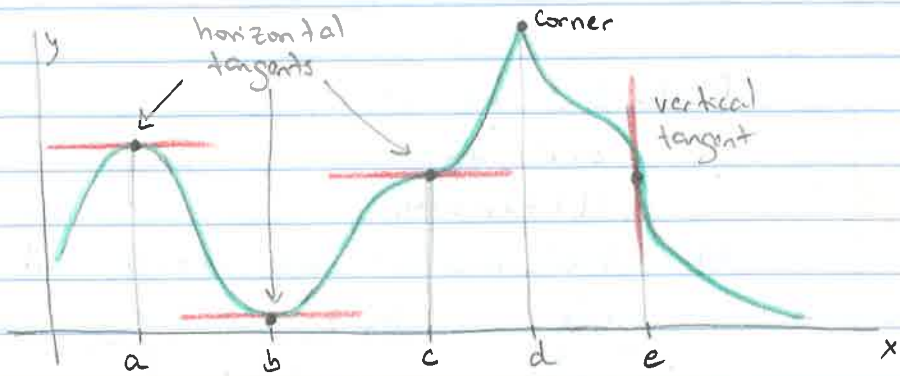


f has a relative min at 0, but $f'(0)$ is not defined.

Potential homework problem: Prove that $f'(0)$ does not exist for $f(x) = |x|$. Show all work.

Def. A critical number of a function f is any x in the domain of f s.t. $f'(x) = 0$ or $f'(x)$ does not exist.

Ex:



Critical points at a, b, c, d, and e.

Formal procedure for finding the relative extrema of a continuous function that is differentiable everywhere except finite values of x: First Derivative Test

First Derivative Test

1) Determine the critical numbers of f.

2) Determine the sign of $f'(x)$ to the left and right of each critical number.

a. $f'(x) > 0$ | $f'(x) < 0$ at c $\Rightarrow$ relative max at c

b. $f'(x) < 0$ | $f'(x) > 0$ at c $\Rightarrow$ relative min at c

c. $f'(x) > 0$ | $f'(x) > 0$ at c

or $f'(x) < 0$ | $f'(x) < 0$ at c

$\Rightarrow$ no relative extremum at c.

Ex: Find the relative max and relative min of

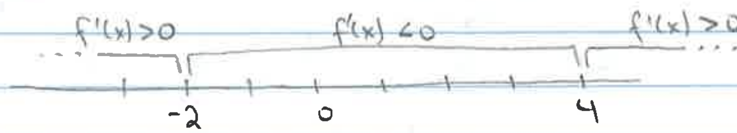
$f(x) = x^3 - 3x^2 - 24x + 32$

$f'(x) = 3x^2 - 6x - 24$

$= 3(x^2 - 2x - 8)$

$= 3(x+2)(x-4)$

$\Rightarrow$ critical numbers at $x = -2$ and $x = 4$



$\Rightarrow$ relative max at $x = -2$

relative min at $x = 4$