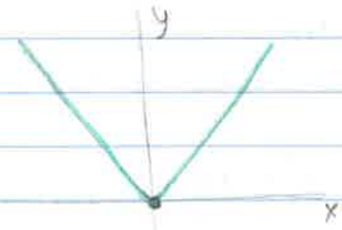
Note: A relative extremum of a function may exist at a point where the derivative does not exist.

Ex:

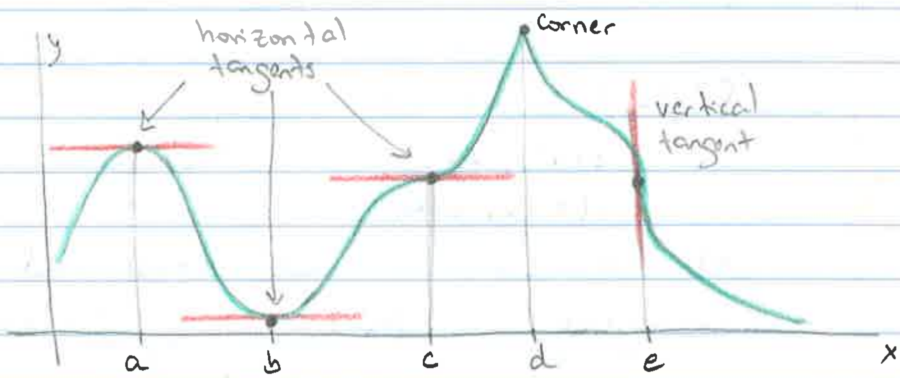


f has a relative min at 0, but f'(0) is not defined.

Potential homework problem: Prove that f'(0) does not exist for f(x) = |x|. Show all work.

Def. A critical number of a function f is any x in the domain of f s.t. f'(x) = 0 or f'(x) does not exist.

Ex:



Critical points at a, b, c, d, and e.

Formal procedure for finding the relative extrema of a continuous function that is differentiable everywhere except finite values of x: First Derivative Test

First Derivative Test

1) Determine the critical numbers of f.

2) Determine the sign of $f'(x)$ to the left and right of each critical number.

a. $f'(x) > 0$ | $f'(x) < 0$ at c $\Rightarrow$ relative max at c

b. $f'(x) < 0$ | $f'(x) > 0$ at c $\Rightarrow$ relative min at c

c. $f'(x) > 0$ | $f'(x) > 0$ at c

or $f'(x) < 0$ | $f'(x) < 0$ at c

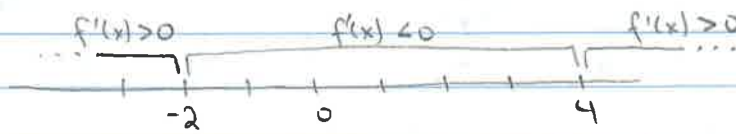
$\Rightarrow$ no relative extremum at c.

Ex: Find the relative max and relative min of

$f(x) = x^3 - 3x^2 - 24x + 32$

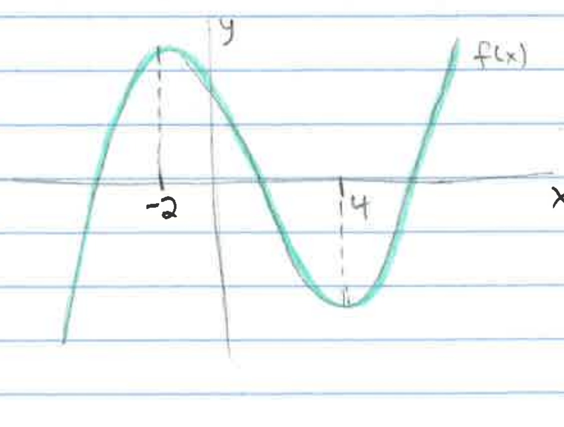
$$f'(x) = 3x^2 - 6x - 24$$
$$= 3(x^2 - 2x - 8)$$
$$= 3(x+2)(x-4)$$

$\Rightarrow$ critical numbers at $x = -2$ and $x = 4$



$\Rightarrow$ relative max at $x = -2$

relative min at $x = 4$



Ex: Find the relative extrema of

$f(x) = x^{2/3}$

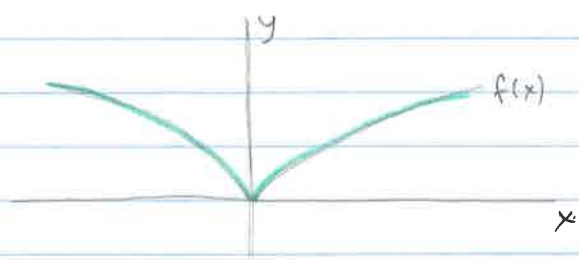
$f'(x) = \frac{2}{3} x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$

$\Rightarrow$ critical number at $x = 0$

... $f'(x) < 0$ | $f'(x) > 0$ ...

0

$\Rightarrow$ relative min at $x = 0$



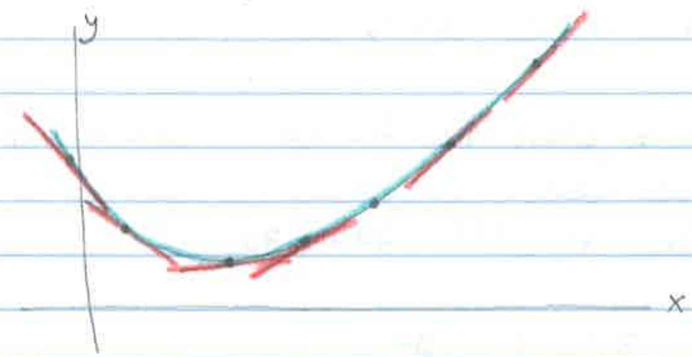
4.2 Applications of the Second Derivative

Determining the Intervals of Concavity

Def. concave upward: 

concave downward: 

Ex: Concave upward:



Notice: slope of the tangent line is increasing.
$\Rightarrow$ The graph f is concave up on (a,b) if f' is increasing on (a,b).

If f' is increasing, then f'' must be positive.
$\Rightarrow$ Thm: The graph f is concave up on (a,b) if f'' is positive on (a,b).

- Likewise, the graph f is concave down on (a,b) if f'' is negative on (a,b).

☆ HW: Prove this.

Def. We say f is $\left\{\begin{array}{l}\text{concave up}\\ \text{concave down}\end{array}\right\}$ at a # c, if there exists an interval (a,b) containing c on which f is $\left\{\begin{array}{l}\text{concave up}\\ \text{concave down}\end{array}\right\}$

Determining the Intervals of Concavity of f

1) Determine the values of x for which f'' is zero or where f'' is not defined, and identify the open intervals determined by these numbers.
2) Determine the sign of f'' in each interval found in Step 1. To do this, compute $f''(c)$, where c is any conveniently chosen test # in the interval.
   a. If $f''(c) > 0$, then the graph of f is concave up on that interval.
   b. If $f''(c) < 0$, then the graph of f is concave down on that interval.

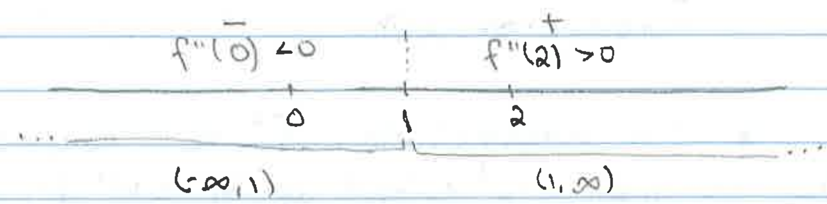
Ex: Determine where the function is concave up and concave down.

1) $f(x) = x^3 - 3x^2 - 24x + 32$

$f'(x) = 3x^2 - 6x - 24$

$f''(x) = 6x - 6 = 6(x-1)$

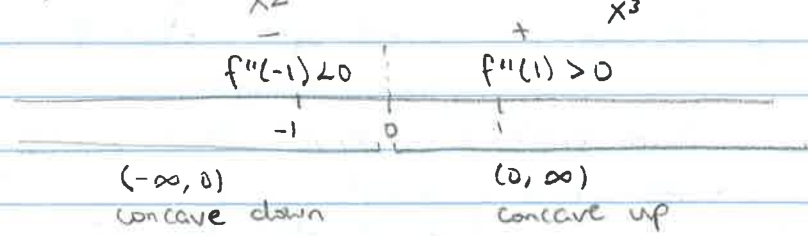
$\Rightarrow f''(x) = 0$ at $x = 1$



$\Rightarrow$ concave up: $(1, \infty)$
concave down: $(-\infty, 1)$

2) $f(x) = x + \frac{1}{x}$

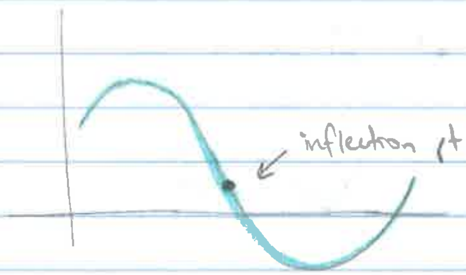
$f'(x) = 1 + \frac{-1}{x^2} \Rightarrow f''(x) = \frac{2}{x^3}$



Inflection Points

Def. A point on the graph of a cts function f where the tangent line exists and where the concavity changes is called a <u>point of inflection</u> or an inflection point

Ex:



Finding Inflection Points

1) Compute $f''(x)$.
2) Determine #s in the domain of f for which $f''(x) = 0$ or $f''(x)$ does not exist.
3) Determine the sign of $f''(x)$ to the left and right of each # c found in Step 2.

   If there is a change in the sign of $f''(x)$ as we move across $x = c$, then $(c, f(c))$ is an inflection point of f.

Ex: Determine the intervals where the graph of the function $f(x) = (x-1)^{5/3}$ is concave up, concave down, and find the inflection points of f.

$f'(x) = \frac{5}{3}(x-1)^{2/3} \quad \Rightarrow \quad f''(x) = \frac{10}{9}(x-1)^{-1/3}$

$\Rightarrow \quad f''(x)$ DNE @ $x = 1$

$-$ $f''(0) < 0$ | $+$ $f''(2) > 0$

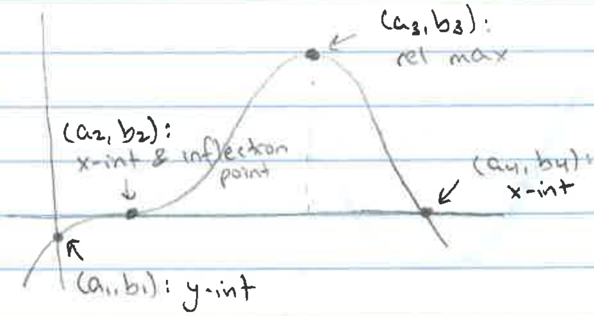
0   1   2

$(-\infty, 1)$: concave down $(1, \infty)$: concave up

inflection pt: $(1, f(1))$

HW. Sketch a graph of the function
$f(x) = x^4 - 4x^3$
without using a calculator. Clearly label
all critical points, inflection points, y-intercept,
and x-intercepts.

Ex:



The Second Derivative Test

Can use 2nd derivative to determine if a critical number is a relative max or relative min of f

1) Compute f' and f''
2) Find all critical #s of f at which $f'(x) = 0$
3) Compute $f''(c)$ for each such critical #
   a. $f''(c) < 0 \Rightarrow$ f has rel max at c
   b. $f''(c) > 0 \Rightarrow$ f has relative min at c
   c. $f''(c) = 0$ or $f''(c)$ does not exist $\Rightarrow$ inconclusive

Note: 2nd derivative test only works for critical points where $f'(x) = 0$. If $f'(x)$ is undefined, we need to use 1st derivative test.

Ex: Determine the relative extrema of the function

$f(x) = x^3 - 3x^2 - 24x + 32$

$f'(x) = 3x^2 - 6x - 24$

$= 3(x^2 - 2x - 8)$

$= 3(x-4)(x+2)$

$\Rightarrow f'(x) = 0$ for $x = 4$ and $x = -2$

$f''(x) = 6x - 6$

$f''(4) = 6(4) - 6 = 24 - 6 = 18 > 0$

$\rightarrow$ relative min at $x = 4$

$f''(-2) = 6(-2) - 6 = -18 < 0$

$\rightarrow$ relative max at $x = -2$

4.3 Curve Sketching

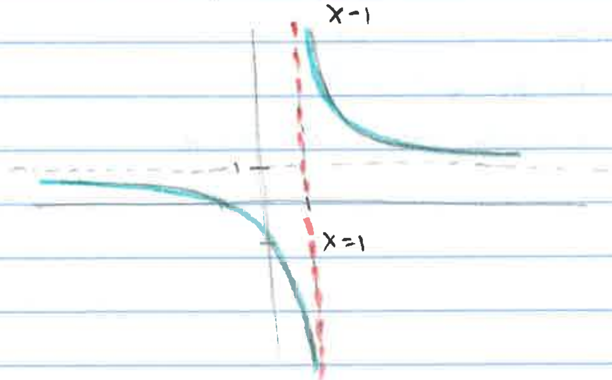
Vertical Asymptotes

Def. The line $x=a$ is a vertical asymptote of f if

$$\lim_{x \to a^+} f(x) = \infty \text{ or } -\infty$$

$$\lim_{x \to a^-} f(x) = \infty \text{ or } -\infty$$

Ex: Consider $f(x) = \dfrac{x+1}{x-1}$



$$\lim_{x \to 1^+} f(x) = \infty$$

$$\lim_{x \to 1^-} f(x) = -\infty$$

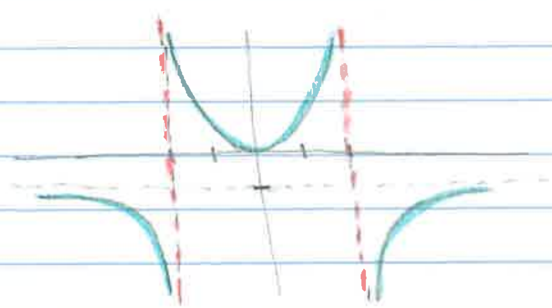
Proposition - If $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function ($P(x)$ and $Q(x)$ are both polynomials), then line $x=a$ is a vertical asymptote of f if $Q(a)=0$ but $P(a) \neq 0$.

Ex: Find the vertical asymptotes of $f(x) = \dfrac{x^2}{4-x^2}$

Want to find the zeros of $Q(x) = 4-x^2$:

$$4-x^2 = 0 \iff (2-x)(2+x) = 0$$

$$\iff x=2 \text{ or } x=-2$$

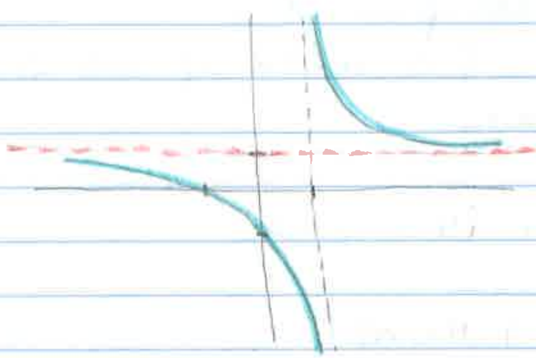


Horizontal Asymptotes

Def. The line $y=b$ is a <u>horizontal asymptote</u> of f if

$$\lim_{x\to\infty} f(x) = b \quad \text{or} \quad \lim_{x\to-\infty} f(x) = b$$

Ex: Consider $f(x) = \dfrac{x+1}{x-1}$



$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 1$$

Ex: Find the horizontal asymptotes of $f(x) = \dfrac{x^2}{4-x^2}$.

$$\lim_{x\to\infty} \frac{x^2}{4-x^2} = -1 \quad \text{and} \quad \lim_{x\to-\infty} \frac{x^2}{4-x^2} = -1.$$

horizontal asymptote: $y=-1$

A Guide to Curve Sketching

1) Determine the domain of f.
2) Find the x- and y- intercepts.
3) Find all horizontal and vertical asymptotes of f.
4) Find the relative extrema of f
5) Determine the intervals where f is increasing and where f is decreasing.
6) Find the inflection points of f
7) Determine where f is concave up and where f is concave down.

Example: Sketch $f(x) = \frac{3(x+2)}{(x-1)^2}$

1) Domain: $(-\infty, 1) \cup (1, \infty)$

2) y-int = 6 $(f(0) = 6)$

   x-int = -2 $(f(-2) = 0)$

3) horizontal asymptotes: $\lim_{x\to\infty} \frac{3(x+2)}{(x-1)^2} = \lim_{x\to\infty} \frac{3x}{3x^2} = 0$

$$\lim_{x\to-\infty} \frac{3(x+2)}{(x-1)^2} = 0$$

vertical asymptotes: $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \infty$

4 & 5) extrema:

$$f'(x) = \frac{3(x-1)^2 - 3(x+2)\cdot 2(x-1)}{(x-1)^{\cancel{4}3}}$$

$$= \frac{3(x-1) - 6(x+2)}{(x-1)^3}$$

$$= \frac{-3(x+5)}{(x-1)^3}$$

$\Rightarrow$ critical pts: $x = 1$, $x = -5$

f'   −   -5   +   1   −

rel min: $(-5, -\frac{1}{4})$