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Multiple Choice

1.(4 pts) Given the cost function:

$$C(x) = x^2 - 3x.$$

Compute the **marginal cost** when $x = 10$.

(a) 7 (b) 10 (c) 700 (d) 17 (e) 70

$C'(x) = 2x - 3$

$\Rightarrow C'(10) = 2(10) - 3 = 20 - 3 = \boxed{17}$

2.(4 pts) Suppose the **elasticity of demand** when the unit price $p = \$100$ is

$$E(100) = \frac{3}{2}.$$

Which **ONE** of the following statement is **TRUE**:

~~(a)~~ The demand when $p = \$100$ is unitary, and decreasing the price will not affect the demand.

(b) The demand when $p = \$100$ is elastic, and decreasing the price will cause an increase in demand.

(c) The demand when $p = \$100$ is elastic, and increasing the price will cause an increase in demand.

~~(d)~~ The demand when $p = \$100$ is inelastic, and decreasing the price will cause a decrease in demand.

~~(e)~~ The demand when $p = \$100$ is inelastic, and increasing the price will cause a decrease in demand.

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3.(4 pts) Given the equation

$$y^2 - x^2 = -8$$

Use implicit differentiation to find $\frac{dy}{dx}$ when $y = -1$ and $x = 3$.

(a) -3 (b) -8 (c) -2 (d) 3 (e) 0

$$\frac{d}{dx}\left(y^2 - x^2 = -8\right)$$

$$\Rightarrow 2y\frac{dy}{dx} - 2x = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y} \qquad \Rightarrow \frac{dy}{dx}(3,-1) = \frac{3}{-1} = \boxed{-3}$$

4.(4 pts) Which **ONE** of the following statement is **TRUE** about $f(x) = x^2 + 1$?

(a) f has a relative minimum at $x = 0$ and $y = 0$.

(b) f has a relative minimum at $x = -1$ and $y = 2$.

(c) f has a relative minimum at $x = 0$ and $y = 1$.

(d) f has a relative maximum at $x = 0$ and $y = 1$.

(e) f has NO relative maximum NOR minimum.

$f'(x) = 2x$

$f'(x) = 0$ when $x = 0$

$f''(x) = 2 > 0$

$\Rightarrow$ function is always concave up

$\Rightarrow$ f has a rel. min when $x = 0$

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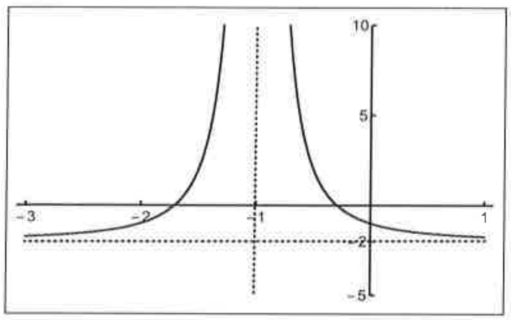
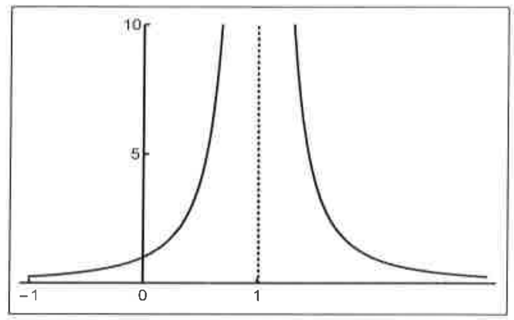
5.(4 pts) g is a function with the following properties:

- g has a **vertical asymptote** at $x = -1$.
- g has a **horizontal asymptote** at $y = 0$.
- g is **increasing** on $(-\infty, -1)$.
- g is **decreasing** on $(-1, \infty)$.
- g is **concave upward** on both intervals $(-\infty, -1)$ and $(-1, \infty)$.

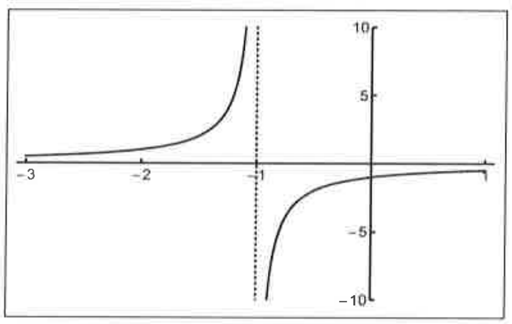
Which **ONE** of the following is the graph of g?



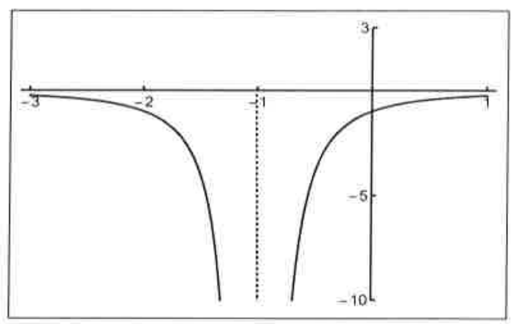
(a)



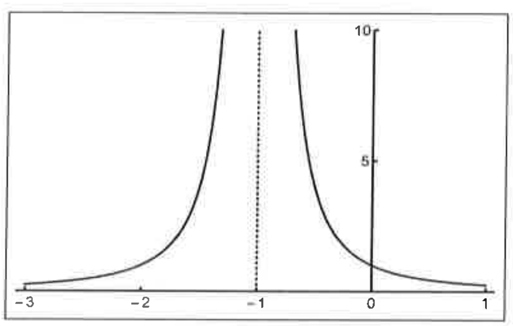
(b)



(e)



(d)



(c)

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Partial Credit

You must show your work on the partial credit problems to receive credit!

6.(8 pts.) The marketing department of Telecon has determined that the demand for their smartphones obeys the relationship:

$$p = -x + 250$$

where p denotes the unit price and x denotes the quantity demanded.

(a) Find the **revenue** function R.

$R(x) = xp$

$R(x) = x(-x + 250)$

$\boxed{R(x) = -x^2 + 250x}$

(b) Compute the **marginal revenue** function.

$\boxed{R'(x) = -2x + 250}$

(c) Compute the **marginal revenue** when $x = 5$.

$R'(5) = -2(5) + 250$

$= -10 + 250$

$= \boxed{240}$

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7.(8 pts.) Given the demand equation:

$$x + \frac{1}{2}p = 9$$

$\Rightarrow x = 9 - \frac{1}{2}p$

$\Rightarrow f(p) = 9 - \frac{1}{2}p$

$\Rightarrow f'(p) = -\frac{1}{2}$

(a) Compute the elasticity of demand when $p = 4$.

$$E(p) = \frac{-p f'(p)}{f(p)}$$

$$= \frac{-p(-\frac{1}{2})}{9 - \frac{1}{2}p} = \frac{\frac{1}{2}p}{9 - \frac{1}{2}p}$$

$$E(4) = \frac{\frac{1}{2}(4)}{9 - \frac{1}{2}(4)} = \boxed{\frac{2}{7}}$$

(b) The demand is *inelastic/unitary/elastic* (circle one) at $p = 4$.

(c) Increasing the price when $p = 4$ by 1% will cause the demand to *increase/decrease* (circle one) by *more than/less than/exactly* (circle one) 1%.

Recall

$$E(p) = -\frac{\%\text{ rate of change of } f \text{ at } p}{\%\text{ rate of change of } p}$$

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9.(8 pts.) (a) Use implicit differentiation to compute $\dfrac{dy}{dx}$ given

$$-x^3 + xy^3 + 11 = 5y^2$$

$$\frac{d}{dx}\left(-x^3+xy^3+11\right) = \frac{d}{dx}\left(5y^2\right)$$

$$\Rightarrow \quad -3x^2 + \left(y^3 + x \cdot 3y^2 \frac{dy}{dx}\right) = 10y\frac{dy}{dx}$$

$$\Rightarrow \quad -3x^2 + y^3 + 3xy^2\frac{dy}{dx} = 10y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx}\left(3xy^2 - 10y\right) = 3x^2 - y^3$$

$$\Rightarrow \quad \boxed{\frac{dy}{dx} = \frac{3x^2 - y^3}{3xy^2 - 10y}}$$

(b) Find the **slope** of the tangent line to the curve above when $x = 2$ and $y = 1$.

$$\left.\frac{dy}{dx}\right|_{(2,1)} = \frac{3(2)^2 - 1^3}{3(2)(1)^2 - 10(1)}$$

$$= \frac{12 - 1}{6 - 10}$$

$$= \boxed{\frac{11}{-4}}$$

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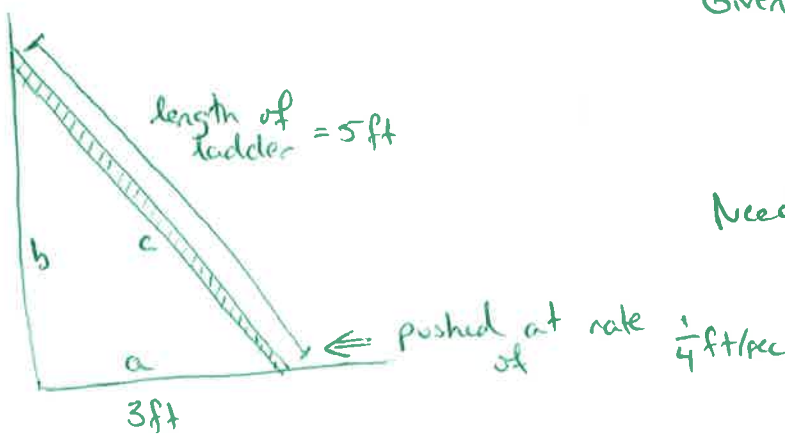
10.(8 pts.) A 5 foot ladder is being pushed against the wall at a rate of $\frac{1}{4}$ ft/sec. How fast is the top of the ladder moving up the wall when the bottom of the ladder is 3 ft away from the wall?



Given: $a = 3\text{ft}$ $\quad \frac{da}{dt} = -\frac{1}{4}$ ← length of a is decreasing

$c = 5\text{ft}$ $\quad \frac{dc}{dt} = 0$

↑ length of ladder does not change

Need: $\frac{db}{dt}$

First, we find b:

$a^2 + b^2 = c^2$

$\Rightarrow 3^2 + b^2 = 5^2$

$\Rightarrow \quad b = \sqrt{5^2 - 3^2}$

$= \sqrt{25 - 9}$

$\Rightarrow b = 4$

Implicit differentiation of Pythagorean equation wrt t:

$\frac{d}{dt}(a^2 + b^2 = c^2)$

$\Rightarrow 2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$

$\Rightarrow 2(3)\left(-\frac{1}{4}\right) + 2(4)\frac{db}{dt} = \cancel{2(5)(0)} \to 0$

$\Rightarrow \frac{db}{dt} = \frac{3/2}{8} = \frac{3}{16}$ $\Rightarrow$ Ladder is moving up the wall at a rate of $\boxed{\frac{3}{16} \text{ ft/sec}}$

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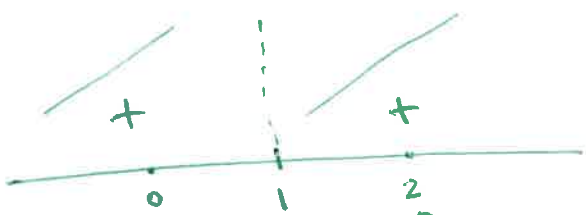
11.(8 pts.) Find the interval of **increasing** and **decreasing** of the function

$$f(x) = \frac{1}{3}x^3 - x^2 + x - 3.$$

$f'(x) = x^2 - 2x + 1$

$= (x-1)^2$

$\Rightarrow$ critical point: $x = 1$



$f'(0) = (0-1)^2 > 0$

$f'(2) = (2-1)^2 > 0$

f is **decreasing** on: none (write none if there is none).

f is **increasing** on: (write none if there is none).

$(-\infty, 1) \cup (1, \infty)$

$\Rightarrow$ everywhere except at $x = 1$

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12.(8 pts.) Find the interval of **concavity** of the function

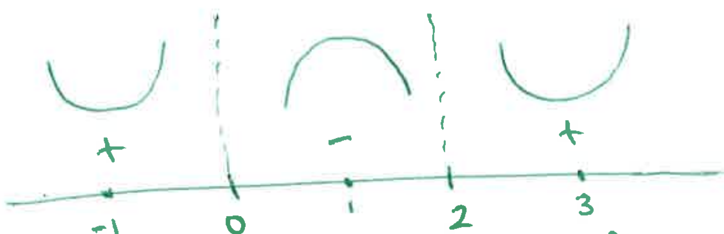
$$g(x) = \frac{1}{4}x^4 - x^3 + x.$$

$g'(x) = x^3 - 3x^2 + 1$

$g''(x) = 3x^2 - 6x$

$= 3x(x-2)$

$\Rightarrow g''(x) = 0$ if $x = 0$ or $x = 2$



$g''(3) = 3(3)(3-2) > 0$

$g''(-1) = 3(-1)(-1-2) > 0$ $g''(1) = 3(1)(1-2) < 0$

g is **concave downward** on: $(0, 2)$ (write none if there is none).

g is **concave upward** on: $(-\infty, 0) \cup (2, \infty)$ (write none if there is none).

g is has **inflection point(s)** at $x =$ $0, 2$ (write none if there is none).

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13.(8 pts.) Find the **vertical asymptote(s)** and the **horizontal asymptote(s)** of

$$f(x) = \frac{3x^2}{x^2-1} = \frac{3x^2}{(x-1)(x+1)}$$

V.A: $(x-1)(x+1) = 0$ if $x=1$ or $x=-1$

$\lim_{x\to -1^-} f(x) = +\infty$ $\qquad$ $\lim_{x\to 1^-} f(x) = -\infty$

$\lim_{x\to -1^+} f(x) = -\infty$ $\qquad$ $\lim_{x\to 1^+} f(x) = +\infty$

$\Rightarrow$ vertical asymptotes at $\boxed{x=1 \text{ and } x=-1}$

HA: $\lim_{x\to\infty} \frac{3x^2}{x^2-1} = \lim_{x\to\infty} \frac{3x^2}{x^2-1} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

$= \lim_{x\to\infty} \frac{3}{1-\frac{1}{x^2}\to 0}$

$= 3$

$\lim_{x\to -\infty} \frac{3x^2}{x^2-1} = \underset{\text{similarly}}{\cdots\cdots} = 3$

$\Rightarrow$ horizontal asymptote at $\boxed{y=3}$

The **vertical asymptote(s)** is: $x =$ <u>1, -1</u> (write none if there is none).

The **horizontal asymptote(s)** is: $y =$ <u>3</u> (write none if there is none).

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14.(8 pts.) Find the **absolute maximum value** and the **absolute minimum value** of the function

$$h(x) = x + \frac{4}{x} \qquad \text{on } [1,4].$$

$h'(x) = 1 - \frac{4}{x^2}$

$1 - \frac{4}{x^2} = 0 \Rightarrow x^2\left(1 - \frac{4}{x^2}\right) = x^2(0)$

$\Rightarrow x^2 - 4 = 0$

$\Rightarrow x = \pm 2$, but $x = -2$ is outside $[1,4]$, so we ignore it.

x	$y = x + \frac{4}{x}$	
1	$1 + \frac{4}{1} = 5$	$\longrightarrow$ absolute max
2	$2 + \frac{4}{2} = 4$	$\longrightarrow$ absolute min
4	$4 + \frac{4}{4} = 5$	$\longrightarrow$ absolute max

The **absolute maximum value** of h is: 5

The **absolute minimum value** of h is: 4

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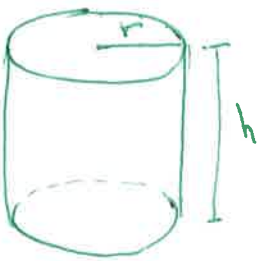
15.(8 pts.) A soup company wants to package its soup in a (hollow) can with a fixed volume of $4\pi\ in^3$. What is the least amount of aluminum (in in^2) needed to create a can with this fixed volume? You should explain why the value you find gives the minimum.

(Recall: The volume of a cylinder with height h and radius r is given by $V = \pi r^2 h$ and its Surface Area is $SA = 2\pi r^2 + 2\pi r h$)



Need to minimize SA.

Have: $V = 4\pi \Rightarrow \cancel{\pi} r^2 h = 4\cancel{\pi}$

$\Rightarrow h = \frac{4}{r^2}$

$\Rightarrow SA = 2\pi r^2 + 2\pi r\left(\frac{4}{r^2}\right)$

$= 2\pi r^2 + \frac{8\pi}{r}$

$SA' = 4\pi r - \frac{8\pi}{r^2} \Rightarrow 4\pi r - \frac{8\pi}{r^2} = 0$

$\Rightarrow 4\pi r = \frac{8\pi}{r^2}$

$\Rightarrow r^3 = \frac{8\pi}{4\pi}$

$\Rightarrow r = \sqrt[3]{2}$

critical point

$SA'' = 4\pi + \frac{16\pi}{r^3}$

SA is concave up for all $r > 0$, so SA has an absolute min at $r = 2^{1/3}$.

minimum amount of aluminum needed

$SA_{min} = 2\pi\left(2^{1/3}\right)^2 + \frac{8\pi}{2^{1/3}}$

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Extra Credit

1.(3 pts.) Deduce the quotient rule from the product rule.

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)\cdot[g(x)]^{-1}\right)$$

$$= \frac{d}{dx}(f(x))\cdot[g(x)]^{-1} + f(x)\cdot \overbrace{\frac{d}{dx}\left([g(x)]^{-1}\right)}^{-1[g(x)]^{-2}\cdot g'(x)}$$

$$= \frac{f'(x)}{g(x)} + \frac{-f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{f'(x)g(x)}{[g(x)]^2} - \frac{f(x)g'(x)}{[g(x)]^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

2.(3 pts.) Approximate the quantity $\sqrt{17}$.

Let $y = f(x) = \sqrt{x}$

$\star$ 16 is nearest perfect square, so take $x_1 = 16$, $x_2 = 17$

$\Rightarrow \Delta x = 1$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2}$$

$$dy = \frac{1}{2\sqrt{x}}\,dx$$

$$\Rightarrow \Delta y \approx \frac{1}{2\sqrt{x}}\Delta x$$

$$= \frac{1}{2\sqrt{16}}(1)$$

$$= \frac{1}{2\cdot 4} = \frac{1}{8}$$

$$\Rightarrow \sqrt{17} - \sqrt{16} = \Delta y \approx \frac{1}{8}$$

$$\Rightarrow \boxed{\sqrt{17} \approx \sqrt{16} + \frac{1}{8}}$$