Solution

MATH 10250 Practice Exam 2 - Extras

1. The weekly demand for a certain product is given by the demand equation

$$p = 20 - 4x$$

where p denotes the wholesale unit price in dollars and x denotes the quantity demanded. The weekly total cost function for manufacturing the products is given by

$$C(x) = -2x^2 + 10x + 12$$

where $C(x)$ denotes the total cost incurred in producing x units.

(a) Find the **revenue** function.

$R(x) = x \cdot p \Rightarrow R(x) = x(20-4x) \Rightarrow \boxed{R(x) = 20x - 4x^2}$

(b) Find the **marginal revenue** function.

$\boxed{R'(x) = 20 - 8x}$

(c) Compute the **marginal revenue** when $x = 2$.

↳ find $R'(2)$

$R'(2) = 20 - 8 \cdot 2$

$\boxed{R'(2) = 4}$

(d) Compute the **marginal cost** function.

$\boxed{C'(x) = -4x + 10}$

(e) Compute the **profit** function.

$P(x) = R(x) - C(x) \Rightarrow P(x) = \overbrace{(20x - 4x^2)}^{R(x)} - \overbrace{(-2x^2 + 10x + 12)}^{C(x)}$

$\Rightarrow P(x) = 20x - 4x^2 + 2x^2 - 10x - 12$

$\Rightarrow \boxed{P(x) = 10x - 2x^2 - 12}$

(f) Compute the **marginal profit** when $x = 2$.

↳ compute $P'(2)$

$P'(x) = 10 - 4x$

$P'(2) = 10 - 4(2) \Rightarrow \boxed{P'(2) = 2}$

formula: $E(p) = \frac{-pf'(p)}{f(p)}$

2. (a) Compute the **elasticity of demand** when $p = 18$ given the demand equation:

$$3x + p = 27$$ → solve for x to get $f(p)$

$\Rightarrow 3x = -p + 27$

$\Rightarrow x = \frac{-p+27}{3} \Rightarrow x = \left(-\frac{1}{3}\right)p + 9$

So, $f(p) = \left(-\frac{1}{3}\right)p + 9 \Rightarrow f'(p) = -\frac{1}{3}$

So, $E(p) = \frac{-p\left(-\frac{1}{3}\right)}{\left(-\frac{1}{3}\right)p + 9} = \frac{p\left(\frac{1}{3}\right)}{\left(-\frac{1}{3}\right)p + 9}$

$E(18) = \frac{18\left(\frac{1}{3}\right)}{\left(-\frac{1}{3}\right)18 + 9} = \frac{6}{-6+9} = \frac{6}{3} = \boxed{2}$

(b) Is the demand inelastic, elastic, or unitary at $p = 18$? Why?

The demand is $\boxed{\text{elastic}}$ at $p = 18$ because $E(18) = 2 > 1$

(c) What will happen to the revenue if you decide to increase the price to 19?

Since the demand at $p = 18$ is elastic, increase the price to 19 will decrease the revenue.

(d) How should you adjust the price in order to increase the revenue?

To increase the revenue, we should decrease the price sightly (maybe to \#17)

3. Given $f(x) = x^3 - 3x^2 - 24x + 32$

(a) Find the interval of **increasing** and **decreasing** of f. List all the **critical numbers** of f.

$f'(x) = 3x^2 - 6x - 24$

$3x^2 - 6x - 24 = 0 \Rightarrow 3(x^2 - 2x - 8) = 0 \Rightarrow 3(x-4)(x+2) = 0$

$\Rightarrow \underbrace{x = -2}$ or $\underbrace{x = 4}$

critical numbers

sign of f': $+$ | $-$ | $+$

-3 $\quad -2 \quad 0 \quad 4 \quad 5$

$f'(-3) = 3(-7)(-1) > 0$ $\qquad f'(0) = -24 < 0$ $\qquad f'(5) = 3(1)(7) > 0$

So,

f is increasing on : $(-\infty, -2) \cup (4, \infty)$
f is decreasing on : $(-2, 4)$
all the critical numbers of f are : $x = -2$ and $x = 4$

(b) Find the **relative maximum** and **relative minimum** of f.

From part (a), we see that f has a relative maximum at $x = -2$ and the relative maximum is $f(-2) = (-2)^3 - 3 \cdot 4 - 24(-2) + 32$

$= -8 - 12 + 48 + 32 = \boxed{60}$

f has a relative minimum at $x = 4$ and the relative maximum is $f(4) = 4^3 - 3 \cdot 4^2 - 24(4) + 32 = 64 - 48 - 96 + 32 = \boxed{-48}$

(c) Find the interval of **concavity** of f. Where is the **inflection point(s)** of f?

$f'(x) = 3x^2 - 6x - 24$

$\Rightarrow f''(x) = 6x - 6$

$6x - 6 = 0 \Rightarrow x = 1$

$f''(0) = -6 < 0$ $\qquad$ $f''(2) = 6 > 0$

So, f is concave downward on: $(-\infty, 1)$

f is concave upward on: $(1, \infty)$

f has an inflection point at $x = 1$, $y = 6$

$(y = f(1) = 1 - 3 - 24 + 32 = -26 + 32 = 6)$

4. Find the **absolute maximum value** and **absolute minimum value** of $g(x) = 2x + \dfrac{8}{x}$ $\boxed{\text{on } [1, 8].}$

$g(x) = 2x + 8x^{-1}$

$g'(x) = 2 - 8x^{-2} = 2 - \dfrac{8}{x^2}$

$2 - \dfrac{8}{x^2} = 0 \Rightarrow x^2\left(2 - \dfrac{8}{x^2}\right) = x^2 0 \Rightarrow 2x^2 - 8 = 0$

$\Rightarrow 2(x^2 - 4) = 0$

$\Rightarrow 2(x-2)(x+2) = 0$

$\Rightarrow \underbrace{x = 2}_{\text{take}}, \underbrace{x = -2}_{\text{outside of } [1,8] \rightarrow \text{ignore}}$

x	$y = 2x + \frac{8}{x}$
1	$2 + 8 = 10$
2	$4 + 4 = 8$ ← abs min
8	$16 + 1 = 17$ ← abs max

So, the abs max value is 17

the abs min value is 8

5. Find the **horizontal asymptote(s)** and **vertical asymptote(s)** of

(a) $f(x) = \dfrac{x^2 + x}{2x + 1}$

HA: $\lim_{x\to\infty} \dfrac{x^2+x}{2x+1} = \lim_{x\to\infty} \dfrac{\frac{x^2}{x^2}+\frac{x}{x^2}}{\frac{2x}{x^2}+\frac{1}{x^2}} = \lim_{x\to\infty} \dfrac{1+\frac{1}{x}}{\frac{2}{x}+\frac{1}{x^2}} = \dfrac{1+0}{0+0} = \dfrac{1}{0} \to$ DNE

Similarly, $\lim_{x\to-\infty} \dfrac{x^2+x}{2x+1} \to$ DNE

So, f has $\boxed{\text{no H.A}}$

VA: $\dfrac{x^2+x}{2x+1} = \dfrac{x(x+1)}{2x+1} \Rightarrow 2x+1 = 0 \Rightarrow x = -\frac{1}{2} \to$ VA is $\boxed{x = -\frac{1}{2}}$

(b) $g(x) = \dfrac{x^2}{4x^2 - 4}$

HA: $\lim_{x\to\infty} \dfrac{x^2}{4x^2-4} = \lim_{x\to\infty} \dfrac{\frac{x^2}{x^2}}{\frac{4x^2}{x^2}-\frac{4}{x^2}} = \lim_{x\to\infty} \dfrac{1}{4-\frac{4}{x^2}} = \dfrac{1}{4-0} = \dfrac{1}{4}$

Similarly, $\lim_{x\to-\infty} \dfrac{x^2}{4x^2-4} = \dfrac{1}{4}$

So, $\boxed{\text{HA is } y = \frac{1}{4}}$

VA: $\dfrac{x^2}{4x^2-4} = \dfrac{x^2}{4(x^2-1)} = \dfrac{x^2}{4(x-1)(x+1)} \to 4(x-1)(x+1) = 0 \to \boxed{\text{VA are } x=1,\ x=-1}$

(c) $h(x) = \dfrac{1}{x^4}$

HA: $\lim_{x\to\infty} \dfrac{1}{x^4} = 0$ and $\lim_{x\to-\infty} \dfrac{1}{x^4} = 0$

So, $\boxed{\text{HA is } y = 0}$

VA: $x^4 = 0 \to \boxed{x = 0 \text{ is the VA}}$

6. The management of Trappee and Sons, producers of the famous TexaPep hot sauce, estimate that their profit from the daily production and sale of x cases of the hot sauce is given by

$$P(x) = -2x^3 + 6x + 40$$

How many cases of hot sauce that the company should produce in order to maximize its profit? What will that maximum profit they can get in one day?

P : profit

x : cases of hot sauce sold. $(x > 0)$

$P(x) = -2x^3 + 6x + 40$

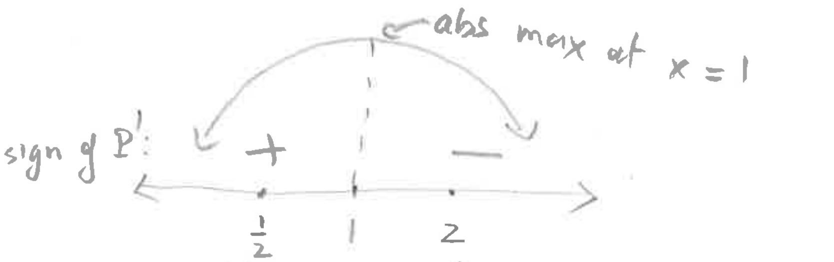
Q1: $x = ?$ so that P is maximum

$P'(x) = -6x^2 + 6$

$-6x^2 + 6 = 0 \Rightarrow -6(x^2 - 1) = 0 \rightarrow -6(x-1)(x+1) = 0 \Rightarrow x = 1,\ x = -1$

not realistic
→ ignore

We want to see if $x = 1$ yields maximum profit:



$P'(\frac{1}{2}) = -\frac{6}{4} + 6 > 0$

$P'(2) = -6(4) + 6 < 0$

So, the company should produce 1 case of hot sauce daily to maximize its profit

Q2: Find P when $x = 1$

$P(1) = -2 \cdot 1^3 + 6 + 40 = \boxed{44}$

using implicit differentiation to

7. Find $\frac{dy}{dx}$ when $x = 1$ and $y = 5$ given

$$x^3 - y^2 = y - 29$$

$$\frac{d}{dx}(x^3 - y^2) = \frac{d}{dx}(y - 29)$$

$$\Rightarrow 3x^2 - 2y\left(\frac{dy}{dx}\right) = \frac{dy}{dx} \quad \Rightarrow 3x^2 = 2y\left(\frac{dy}{dx}\right) + \left(\frac{dy}{dx}\right)$$

$$\Rightarrow 3x^2 = \left(\frac{dy}{dx}\right)[2y + 1]$$

$$\Rightarrow \frac{3x^2}{2y+1} = \frac{dy}{dx} \quad (x=1,\ y=5)$$

$$\Rightarrow \frac{dy}{dx} = \frac{3}{10+1} = \boxed{\frac{3}{11}}$$

8. Suppose the quantity demanded weekly of a certain commodity is related to its unit price by the equation

$$x^2 + 2p = 40$$

where p is the price per unit (in dollars) and x is the quantity demanded. How fast is the price changing weekly when the the demanded quantity is 5 and the quantity demanded is decreasing at the rate 3 units per week?

x = quantity demanded $\rightarrow$ depends on t

p = unit price $\rightarrow$ depends on t

Ask $\frac{dp}{dt} = ?$ when $x = 5$ and $\frac{dx}{dt} = -3$

$$x^2 + 2p = 40$$

$$\Rightarrow \frac{d}{dt}(x^2 + 2p) = \frac{d}{dt}(40)$$

$$\Rightarrow 2x\left(\frac{dx}{dt}\right) + 2\left(\frac{dp}{dt}\right) = 0 \quad \left(x = 5,\ \frac{dx}{dt} = -3\right)$$

$$\Rightarrow 2(5)(-3) + 2\left(\frac{dp}{dt}\right) = 0$$

$$\Rightarrow 2\left(\frac{dp}{dt}\right) = 30$$

$$\Rightarrow \frac{dp}{dt} = 15$$

So, the price is increasing at the rate of $\boxed{15}$ dollars/week