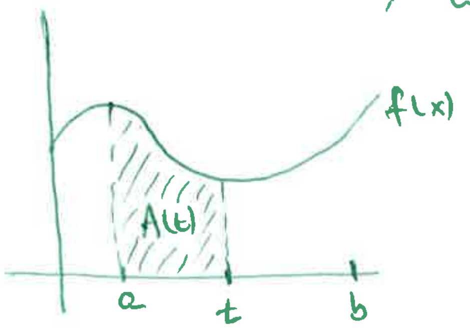
Proof of the Fundamental Theorem of Calculus

Let f be a nonneg function on $[a,b]$

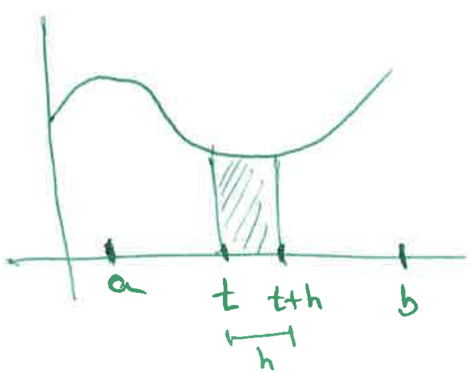
Let $A(t)$ denote the area of the region under $f(x)$ from $x=a$ to $x=t$, where $a \le t \le b$



Let $h>0$ be a small number. Then $A(t+h)$ is the area under $f(x)$ from $x=a$ to $x=t+h$. So,

$$A(t+h) - A(t)$$

is the area under $f(x)$ from $x=t$ to $x=t+h$,



We can approximate the area of this region by the area of a rectangle w/ width h and height $f(t)$:

$$A(t+h) - A(t) \approx h \cdot f(t)$$

$$\Rightarrow f(t) \approx \frac{A(t+h) - A(t)}{h}$$

and the smaller h is, the better the approximation.

To get an equality, we take the limit as $h \to 0$:

$$\lim_{h \to 0} f(t) = \lim_{h \to 0} \frac{A(t+h) - A(t)}{h}$$

$$\Rightarrow \quad f(t) = A'(t)$$

so the area function $A(t)$ is an antiderivative of f.

Thus,

$$A(t) = F(t) + C,$$

where $F(t)$ is an antiderivative of $f(t)$.

Note that $A(a) = 0$

$$\Rightarrow A(a) = F(a) + C = 0$$

$$\Rightarrow \quad C = -F(a) \qquad \Rightarrow \boxed{A(t) = F(t) - F(a)}$$

Now, since the area of the region under $f(x)$ from $x=a$ to $x=b$ is $A(b)$, then

$$A(b) = F(b) - F(a).$$

Since the area of the region is $\int_a^b f(x)\,dx$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$