Proof that $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

$$
\begin{aligned}
\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x &= e^{\ln\left[\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]} \\
&= e^{\lim\limits_{x\to\infty} x\ln\left(1+\frac{1}{x}\right)} \\
&= e^{\lim\limits_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}}} \\
&= e^{\lim\limits_{x\to\infty} \frac{\frac{1}{1+\frac{1}{x}}\cdot\frac{-1}{x^2}}{\frac{-1}{x^2}}}, \quad \text{using L'Hospital's rule} \\
&= e^{\lim\limits_{x\to\infty} \frac{1}{1+\frac{1}{x}}} \\
&= e
\end{aligned}
$$