MATH 10250 Homework 2

1. Deduce the quotient rule from the product rule.

   Hint: Express $\dfrac{f(x)}{g(x)}$ as $f(x)[g(x)]^{-1}$ and differentiate the second. (Don't forget the Chain Rule here!)

   Solution:

$$\begin{aligned}
\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) &= \frac{d}{dx}\left(f(x)[g(x)]^{-1}\right) \\
&= \frac{d}{dx}\Big(f(x)\Big)\cdot[g(x)]^{-1} + f(x)\cdot\frac{d}{dx}\Big([g(x)]^{-1}\Big) \\
&= \frac{f'(x)}{g(x)} + f(x)\Big(-1[g(x)]^{-2}g'(x)]\Big) \\
&= \frac{f'(x)}{g(x)} + \frac{-f(x)g'(x)}{[g(x)]^2} \\
&= \frac{f'(x)g(x)}{[g(x)]^2} - \frac{f(x)g'(x)}{[g(x)]^2} \\
&= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.
\end{aligned}$$

2. Let $f(x)$ be some function. What is the difference between saying "$\lim\limits_{x\to a} f(x)$ exists" and "$f(x)$ is continuous at $x = a$"? Give an example of a function where the first statement is true, but the second is not. (A sketch of the function is fine.)

   Solution: Saying "$\lim\limits_{x\to a} f(x)$ exists" is equivalent to saying $\lim\limits_{x\to a^-} f(x)$ and $\lim\limits_{x\to a^+} f(x)$ both exist and are equal. Now, "$f(x)$ is continuous at $x = a$" means that $\lim\limits_{x\to a} f(x)$ exists AND $f(a)$ exists such that $\lim\limits_{x\to a} f(x) = f(a)$. So, "$f(x)$ is continuous at $x = a$" implies "$\lim\limits_{x\to a} f(x)$ exists", but the reverse is not true.

   Consider the function $f(x) = \dfrac{(x-1)(x+3)}{x+3}$. This function is not continuous at $x = -3$ (since $f(-3)$ is not defined), but $\lim\limits_{x\to -3} f(x)$ does exist and is -4.