AVERAGE INTENSITY OF THE DISTRIBUTION OF COMPLEX ZEROS OF A CLASS OF RANDOM POLYNOMIALS

ETHAN ADDISON*, ANDREW LEDOAN, SPENCER SMITH*, AND RICHARD VRADENBURGH*

Abstract. We establish a formula for the average intensity of the distribution of complex zeros of the random polynomial $\eta_0\omega_0 + \eta_1\omega_1 z + \eta_2\omega_2 z^2 + \cdots + \eta_{n-1}\omega_{n-1}z^{n-1}$ with $z \in \mathbb{C}$ and for any sequence of real constants ω_j and standard normal independent random variables η_j. We further obtain the limiting behavior of this intensity function as n tends to infinity for various sequences $\{\omega_j\}_{j=0}^{n-1}$.

1. Introduction and statement of results

Let

$$P_n(z) = \eta_0\omega_0 + \eta_1\omega_1 z + \eta_2\omega_2 z^2 + \cdots + \eta_{n-1}\omega_{n-1}z^{n-1},$$

where $\{\eta_j\}_{j=0}^{\infty}$ is a sequence of independent random variables and $\{\omega_j\}_{j=0}^{n-1}$ is a sequence of real constants. For $\omega_j = 1$ and η_j and z real, there are many asymptotic estimates for the expected number of real zeros of $P_n(z)$. However, little is known about the complex zeros of $P_n(z)$. In 1995 Shepp and Vanderbei [8] initiated the study of the intensity of complex zeros of $P_n(z)$ with $\omega_j = 1$ and for real standard normal random coefficients η_j. Denoting by $\nu_n(\Omega)$ the number of complex zeros in any region $\Omega \subset \mathbb{C}$ of $P_n(z)$, where $z \in \mathbb{C}$, Shepp and Vanderbei showed that the values of the intensity functions h_n and g_n are given by

$$\mathbb{E}[\nu_n(\Omega)] = \int_{\Omega} h_n(x,y)\,dx\,dy + \int_{\Omega\cap\mathbb{R}} g_n(x)\,dx. \tag{1}$$

In this paper, we employ Shepp and Vanderbei's method to study the intensity of complex zeros of $P_n(z)$ for any sequence of real constants ω_j and standard normal random coefficients η_j. We present our main result in the following theorem.

Theorem 1. *For h_n and g_n given in* (1), *we have*

$$h_n(z) = \frac{B_{2,n}(B_{0,n}^2 - |A_{0,n}|^2) + B_{1,n}(A_{0,n}\bar{A}_{1,n} + \bar{A}_{0,n}A_{1,n}) - B_{0,n}(B_{1,n}^2 + |A_{1,n}|^2)}{\pi|z|^2(B_{0,n}^2 - |A_{0,n}|^2)^{3/2}}$$

and

$$g_n(z) = \frac{\sqrt{B_{0,n}B_{2,n} - B_{1,n}^2}}{\pi|z|B_{0,n}},$$

where

$$A_{k,n}(z) = \sum_{j=0}^{n-1} j^k\omega_j^2 z^{2j}, \quad k = 0, 1,$$

and

$$B_{k,n}(z) = \sum_{j=0}^{n-1} j^k\omega_j^2 |z|^{2j}, \quad k = 0, 1, 2.$$

The real zeros of random polynomials with $\omega_j = 1$, real coefficients, and n large are clustered about ± 1. Hence, it is of interest to study the behavior of h_n and g_n as $n \to \infty$. Our choices for ω_j are motivated by the works of Littlewood and Offord [5], [6], [7].

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Corollary 1. *With h_n and g_n as given in Theorem 1, the following results hold.*

(i) *If* $\omega_j = \sqrt{\binom{n-1}{j}/(j+1)}$, *for* $|z| > 0$ *and* n *sufficiently large*

$$h_n(z) \sim \frac{n}{\pi(1+|z|^2)^2}$$

and

$$g_n(z) \sim \frac{\sqrt{n}}{\pi(1+z^2)}.$$

(ii) *If* $\omega_j = \sqrt{j}$, *then*

$$\lim_{n\to\infty} h_n(z) = \begin{cases} \dfrac{|u|^6 v - q^4|u|^2 v - p^2|u|^6 - q^6|t|^2 + 2p^2q^6}{\pi|z|^2 q^2(|u|^4 - q^4)^{3/2}}, & \text{if } 0 < |z| < 1; \\ \dfrac{q^3(z^2\bar{u} + \bar{z}^2 u) - pq^2|u|^2 - |z|^2 q^4 + |u|^4}{\pi q^2|u|(|u|^2 - q^2)^{3/2}}, & \text{if } |z| > 1; \end{cases}$$

where

$$p = 1 + |z|^2, \quad q = 1 - |z|^2, \quad t = 1 + z^2, \quad u = 1 - z^2, \quad v = |z|^4 + 4|z|^2 + 1,$$

and

$$\lim_{n\to\infty} g_n(z) = \begin{cases} \dfrac{\sqrt{2}}{\pi(1-z^2)}, & \text{if } 0 < |z| < 1; \\ \dfrac{1}{\pi(z^2-1)}, & \text{if } |z| > 1. \end{cases}$$

(iii) *If* $\omega_j = 1/\sqrt{j!}$, *then*

$$\lim_{n\to\infty} h_n(z) = \frac{\exp(\mathrm{Im}(z)^2)\sinh(2\mathrm{Im}(z)^2) - 2\mathrm{Im}(z)^2\exp(-\mathrm{Im}(z)^2)}{\pi\sqrt{2}(\sinh(2\mathrm{Im}(z)^2))^{3/2}}$$

and

$$\lim_{n\to\infty} g_n(z) = \frac{1}{\pi}.$$

Shepp and Vanderbei's method is applicable to the case of complex coefficients. Our second result and its consequences can be summarized as follows.

Theorem 2. *Let* $\eta_j = \alpha_j + i\beta_j$, *where* $\{\alpha_j\}_{j=0}^{n-1}$ *and* $\{\beta_j\}_{j=0}^{n-1}$ *are sequences of independent standard normal random variables. For* h_n *given in* (1), *we have*

$$h_n(z) = \frac{B_{0,n}B_{2,n} - B_{1,n}^2}{\pi|z|^2 B_{0,n}^2}.$$

Corollary 2. *With the assumption of Theorem 2, the following results hold.*

(i) *If* $\omega_j = \sqrt{\binom{n-1}{j}/(j+1)}$, *for* n *sufficiently large*

$$h_n(z) \sim \frac{n}{\pi(1+|z|^2)^2}.$$

(ii) *If* $\omega_j = \sqrt{j}$, *then*

$$\lim_{n\to\infty} h_n(z) = \begin{cases} \dfrac{2}{\pi(1-|z|^2)^2}, & \text{if } 0 < |z| < 1; \\ \dfrac{1}{\pi(1-|z|^2)^2}, & \text{if } |z| > 1. \end{cases}$$

(iii) *If* $\omega_j = 1/\sqrt{j!}$, *then*

$$\lim_{n\to\infty} h_n(z) = \frac{1}{\pi}.$$

2. Proof of Theorem 1

Suppose that $\partial\Omega$ intersects the real axis at most finitely many points only and that z does not lie on the real axis. Since $P_n(z)$ is analytic within and on Ω, $P_n(z)$ does not vanish on Ω, and $P_n(z)$ does not have any poles, Cauchy's principle of the argument gives

$$\nu_n(\Omega) = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{P_n'(z)}{P_n(z)}\,dz,$$

where the integral is taken over $\partial\Omega$ in the positive sense. Taking expectations, using Fubini's theorem to interchange expectation and integral (the justification of which is tedious but achievable), and multiplying inside the expectation by z and dividing outside by z, we obtain

$$\mathbb{E}[\nu_n(\Omega)] = \frac{1}{2\pi i}\int_{\partial\Omega}\mathbb{E}\left[\frac{P_n'(z)}{P_n(z)}\right]dz = \frac{1}{2\pi i}\int_{\partial\Omega}\frac{1}{z}\mathbb{E}\left[\frac{zP_n'(z)}{P_n(z)}\right]dz.$$

For the sake of brevity, we let

$$a_j = \mathrm{Re}(z^j), \quad b_j = \mathrm{Im}(z^j), \quad c_j = \mathrm{Im}(jz^j) = ja_j, \quad d_j = \mathrm{Im}(jz^j) = jb_j,$$

and decompose $P_n(z)$ and $zP_n'(z)$ into their real and imaginary parts to obtain

$$P_n(z) = \xi_1 + i\xi_2, \quad zP_n'(z) = \xi_3 + i\xi_4,$$

where

$$\xi_1 = \sum_{j=0}^{n-1}\eta_j\omega_j a_j, \quad \xi_2 = \sum_{j=0}^{n-1}\eta_j\omega_j b_j, \quad \xi_3 = \sum_{j=0}^{n-1}\eta_j\omega_j c_j, \quad \xi_4 = \sum_{j=0}^{n-1}\eta_j\omega_j d_j.$$

The four random variables ξ_1, ξ_2, ξ_3, and ξ_4 are correlated, which can be represented in terms of four standard normal random variables.

For brevity's sake, we let

$$\xi = [\ \xi_1 \quad \xi_2 \quad \xi_3 \quad \xi_4\ ]^T, \quad \omega = [\ \omega_0 \quad \omega_1 \quad \dots \quad \omega_{n-1}\ ]^T, \quad a = [\ a_0 \quad a_1 \quad \dots \quad a_{n-1}\ ]^T,$$
$$b = [\ b_0 \quad b_1 \quad \dots \quad b_{n-1}\ ]^T, \quad c = [\ c_0 \quad c_1 \quad \dots \quad c_{n-1}\ ]^T, \quad d = [\ d_0 \quad d_1 \quad \dots \quad d_{n-1}\ ]^T.$$

Using the shorthand notation $\otimes : M^{m\times n}\times M^{m\times n}\longrightarrow M^{m\times n}$ defined by $\phi_{ij}\otimes\omega_{ij}\longmapsto\phi_{ij}\omega_{ij}$ and the fact that the covariance matrix $\mathrm{Cov}(\xi) = \mathbb{E}(\xi\xi^T)$ can be expressed in terms of the Cholesky factor L as $\mathrm{Cov}(\xi) = LL^T$, where $L = [l_{ij}]$ for $i \geq j$ and $L = 0$ for $i < j$, we obtain the matrix equation

$$\begin{aligned}\mathrm{Cov}(\xi) &= \begin{bmatrix} (\omega\otimes a)^T(\omega\otimes a) & (\omega\otimes a)^T(\omega\otimes b) & (\omega\otimes a)^T(\omega\otimes c) & (\omega\otimes a)^T(\omega\otimes d)\\ (\omega\otimes b)^T(\omega\otimes a) & (\omega\otimes b)^T(\omega\otimes b) & (\omega\otimes b)^T(\omega\otimes c) & (\omega\otimes b)^T(\omega\otimes d)\\ (\omega\otimes c)^T(\omega\otimes a) & (\omega\otimes c)^T(\omega\otimes b) & (\omega\otimes c)^T(\omega\otimes c) & (\omega\otimes c)^T(\omega\otimes d)\\ (\omega\otimes d)^T(\omega\otimes a) & (\omega\otimes d)^T(\omega\otimes b) & (\omega\otimes d)^T(\omega\otimes c) & (\omega\otimes d)^T(\omega\otimes d)\end{bmatrix}\\ &= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} & l_{11}l_{41}\\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} & l_{21}l_{41} + l_{22}l_{42}\\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 & l_{31}l_{41} + l_{32}l_{42} + l_{33}l_{43}\\ l_{41}l_{11} & l_{41}l_{21} + l_{42}l_{22} & l_{41}l_{31} + l_{42}l_{32} + l_{43}l_{33} & l_{41}^2 + l_{42}^2 + l_{43}^2 + l_{44}^2\end{bmatrix}.\end{aligned}$$

The corresponding matrix elements can be equated as sixteen scalar equations. Of these, ten are independent, due to symmetry, and can be solved in sequence, making use of previous results. We will only need to consider the following seven independent scalar equations:

$$\begin{aligned}&l_{11}^2 = (\omega\otimes a)^T(\omega\otimes a),\\ &l_{21}l_{11} = (\omega\otimes b)^T(\omega\otimes a), \quad l_{21}^2 + l_{22}^2 = (\omega\otimes b)^T(\omega\otimes b),\\ &l_{31}l_{11} = (\omega\otimes c)^T(\omega\otimes a), \quad l_{31}l_{21} + l_{32}l_{22} = (\omega\otimes c)^T(\omega\otimes b),\\ &l_{41}l_{11} = (\omega\otimes d)^T(\omega\otimes a), \quad l_{41}l_{21} + l_{42}l_{22} = (\omega\otimes d)^T(\omega\otimes b).\end{aligned}$$

It is easy to see that the solutions using prior results are

$$l_{11} = \frac{(\omega \otimes a)^T(\omega \otimes a)}{\sqrt{(\omega \otimes a)^T(\omega \otimes a)}},$$

$$l_{21} = \frac{(\omega \otimes b)^T(\omega \otimes a)}{\sqrt{(\omega \otimes a)^T(\omega \otimes a)}}, \quad l_{22} = \frac{(\omega \otimes a)^T(\omega \otimes a)(\omega \otimes b)^T(\omega \otimes b) - ((\omega \otimes b)^T(\omega \otimes a))^2}{\sqrt{(\omega \otimes a)^T(\omega \otimes a)}R},$$

$$l_{31} = \frac{(\omega \otimes c)^T(\omega \otimes a)}{\sqrt{(\omega \otimes a)^T(\omega \otimes a)}}, \quad l_{32} = \frac{(\omega \otimes a)^T(\omega \otimes a)(\omega \otimes c)^T(\omega \otimes b) - (\omega \otimes c)^T(\omega \otimes a)(\omega \otimes b)^T(\omega \otimes a)}{\sqrt{(\omega \otimes a)^T(\omega \otimes a)}R},$$

$$l_{41} = \frac{(\omega \otimes d)^T(\omega \otimes a)}{\sqrt{(\omega \otimes a)^T(\omega \otimes a)}}, \quad l_{42} = \frac{(\omega \otimes a)^T(\omega \otimes a)(\omega \otimes d)^T(\omega \otimes b) - (\omega \otimes d)^T(\omega \otimes a)(\omega \otimes b)^T(\omega \otimes a)}{\sqrt{(\omega \otimes a)^T(\omega \otimes a)}R},$$

where

$$R = \sqrt{(\omega \otimes a)^T(\omega \otimes a)(\omega \otimes b)^T(\omega \otimes b) - ((\omega \otimes b)^T(\omega \otimes a))^2}.$$

Using the Cholesky factor L, we obtain the matrix equation

$$\xi \stackrel{\mathrm{D}}{=} L\zeta,$$

where

$$\zeta = [\ \zeta_1 \quad \zeta_2 \quad \zeta_3 \quad \zeta_4\ ]^T$$

is a vector of four independent standard normal random variables, and the symbol $\stackrel{\mathrm{D}}{=}$ is used to denote equality in distribution, for

$$\mathrm{Cov}(\xi) = \mathbb{E}(\xi\xi^T) = L\mathbb{E}(\zeta\zeta^T)L^T = \mathbb{E}(L\zeta\zeta^T L) = \mathrm{Cov}(\zeta).$$

Hence,

$$\frac{zP_n'(z)}{P_n(z)} = \frac{\zeta_3 + i\zeta_4}{\zeta_1 + i\zeta_2} \stackrel{\mathrm{D}}{=} \frac{(l_{31} + il_{41})\zeta_1 + (l_{32} + il_{42})\zeta_2 + (l_{33} + il_{43})\zeta_3 + il_{44}\zeta_4}{l_{11}\zeta_1 + il_{22}\zeta_2}.$$

Taking expectations, making full use of the independence of the standard normal random variables $\zeta_1, \zeta_2, \zeta_3$, and ζ_4, and putting

$$\alpha = l_{31} + il_{41}, \quad \beta = l_{32} + il_{42}, \quad \gamma = l_{11} + il_{21}, \quad \delta = il_{22},$$

we obtain

$$F(z) = \mathbb{E}\left(\frac{\alpha\zeta_1 + \beta\zeta_2}{\gamma\zeta_1 + \delta\zeta_2}\right) = \frac{\alpha}{\delta}\mathbb{E}\left[\frac{\zeta_1}{(\gamma/\delta)\zeta_1 + \zeta_2}\right] + \frac{\beta}{\gamma}\mathbb{E}\left[\frac{\zeta_2}{\zeta_1 + (\delta/\gamma)\zeta_2}\right].$$

Interchanging the roles of ζ_1 and ζ_2 and using the complex-valued function $f(w)$ defined on $\mathbb{C} - \mathbb{R}$ by

$$f(w) = \mathbb{E}\left(\frac{\zeta_1}{w\zeta_1 + \zeta_2}\right),$$

we find that

$$F(z) = \frac{\alpha}{\delta}f\left(\frac{\gamma}{\delta}\right) + \frac{\beta}{\gamma}f\left(\frac{\delta}{\gamma}\right).$$

We now examine the function $f(w)$.

We use the transformation $r = \sqrt{x^2 + y^2} > 0$ and $\phi = \arctan(y/x)$ with $-\pi < \phi \leq \pi$. The Jacobian is computed as

$$\det\left(\frac{\partial(r,\phi)}{\partial(x,y)}\right) = \begin{vmatrix} x/r & y/r \\ -y/r^2 & x/r^2 \end{vmatrix} = \frac{1}{r},$$

and the unique inverse transformation is $x = r\cos\phi$ and $y = r\sin\phi$. Since ζ_1 and ζ_2 are independent standard normal random variables given by

$$f_{\zeta_1\zeta_2}(x,y) = \frac{1}{2\pi}e^{-(x^2+y^2)/2},$$

we have

$$f_{R\Phi}(r,\phi) = \frac{1}{|\det(\partial(r,\phi)/\partial(x,y))|}f_{\zeta_1,\zeta_2}(x,y) = \frac{1}{2\pi}re^{-r^2/2}.$$

On noting that

$$f_R(r) = \int_{-\pi}^{\pi} f_{R\Phi}(r,\phi)\, d\phi = re^{-r^2/2}$$

and

$$f_\Phi(\phi) = \int_0^\infty f_{R\Phi}(r,\phi)\, dr = \frac{1}{2\pi},$$

we have

$$f_{R\Phi}(r,\phi) = f_R(r) f_\Phi(\phi).$$

Hence, the random variables R and Φ are independent.

We use the formula for change of variables in a double integral (see Buck [2], Theorem 10, page 488), convert to polar coordinates, and use the trigonometric substitution $u = \tan\phi$ with $du = (1+u^2)\, d\phi$, to compute that

$$\begin{aligned}
f(w) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{x}{wx+y}\right) f_{\zeta_1\zeta_2}(x,y)\, dx\, dy \\
&= \int_{-\pi}^{\pi}\int_0^\infty \left(\frac{1}{w+\tan\phi}\right) \frac{1}{|\det[\partial(r,\phi)/\partial(x,y)]|} f_{\zeta_1\zeta_2}(r\cos\phi, r\sin\phi)\, dr\, d\phi \\
&= \int_{-\pi}^{\pi}\int_0^\infty \left(\frac{1}{w+\tan\phi}\right) f_{R\Phi}(r,\phi)\, dr\, d\phi \\
&= \int_{-\pi}^{\pi}\int_0^\infty \left(\frac{1}{w+\tan\phi}\right) f_R(r) f_\Phi(\phi)\, dr\, d\phi \\
&= \int_{-\pi}^{\pi} \left(\frac{1}{w+\tan\phi}\right) f_\Phi(\phi)\, d\phi \int_0^\infty f_R(r)\, dr \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{w+\tan\phi}\, d\phi \\
&= \frac{1}{2\pi}\left(\int_{-\pi}^{-\pi/2} + \int_{-\pi/2}^{\pi/2} + \int_{\pi/2}^{\pi}\right) \frac{1}{w+\tan\phi}\, d\phi \\
&= \frac{1}{2\pi}\left(\int_0^\infty + \int_{-\infty}^{\infty} + \int_{-\infty}^{0}\right) \frac{1}{(w+u)(1+u^2)}\, du \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{(w+u)(u-i)(u+i)}\, du \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} g(u)\, du,
\end{aligned}$$

where

$$g(z) = \frac{1}{(w+z)(z-i)(z+i)}$$

has a simple pole at $z = i$ with residue $\mathrm{Res}(g(z); i) = 1/(2i(w+i))$ and at $z = -i$ with residue $\mathrm{Res}(g(z); -i) = 1/(2i(w-i))$. When $\mathrm{Im}(w) > 0$, we take $\rho > 0$ such that the simple pole at the point $z = i$ is enclosed by the contour $\Gamma^+ = [-\rho, \rho] \cup C_\rho^+$, defined in the positive sense with C_ρ^+ parametrized by $z = \rho e^{i\theta}$ with $0 \le \theta \le \pi$. Thus,

$$\begin{aligned}
\int_{\Gamma^+} \frac{1}{(w+z)(z-i)(z+i)}\, dz &= \int_{-\rho}^{\rho} \frac{1}{(w+x)(1+x^2)}\, dx + \int_{C_\rho^+} \frac{1}{(w+z)(z-i)(z+i)}\, dz \\
&= 2\pi i\, \mathrm{Res}(g(z); i) = \frac{\pi}{w+i}.
\end{aligned}$$

Since

$$\left| \int_{C_\rho^+} \frac{1}{(w+z)(z-i)(z+i)}\, dz \right| \le \int_0^\pi \frac{\rho}{\rho^2-1}\, d\theta = \frac{\pi\rho}{\rho^2-1},$$

with the far left-hand side going to zero as ρ approaches infinity, we see that

$$\int_{-\infty}^{\infty} \frac{1}{(w+x)(1+x^2)\,dx} = \frac{\pi}{w+i}.$$

Hence,

$$f(w) = \frac{1}{w+i}$$

when $\mathrm{Im}(w) > 0$. A similar calculation with $\Gamma^- = [-\rho, \rho] \cup C_\rho^-$ defined in the positive sense and C_ρ^- parametrized by $z = \rho e^{-i\theta}$, with $0 \le \theta \le \pi$, yields

$$f(w) = \frac{1}{w-i}$$

when $\mathrm{Im}(w) < 0$.

Returning to F, we observe that the point γ/δ, given by

$$\frac{\gamma}{\delta} = \frac{l_{11} + il_{21}}{il_{22}} = \frac{l_{21}}{l_{22}} - i\frac{l_{11}}{l_{22}},$$

lies in the lower half-plane, so that its reciprocal, δ/γ, lies in the upper half-plane. We have

$$\begin{aligned}
F(z) &= \frac{\alpha}{\delta}\left(\frac{1}{\gamma/\delta - i}\right) + \frac{\beta}{\gamma}\left(\frac{1}{\delta/\gamma + i}\right) = \frac{i\alpha + \beta}{\delta + i\gamma} = \frac{l_{32} - l_{41} + i(l_{31} + l_{42})}{-l_{21} + i(l_{11} + l_{22})} \\
&= \Bigg[\frac{(\omega\otimes a)^T(\omega\otimes a)(\omega\otimes c)^T(\omega\otimes b) - (\omega\otimes c)^T(\omega\otimes a)(\omega\otimes b)^T(\omega\otimes a)}{\sqrt{(\omega\otimes a)^T(\omega\otimes a)}R} - \frac{(\omega\otimes d)^T(\omega\otimes a)}{\sqrt{(\omega\otimes a)^T(\omega\otimes a)}} \\
&\quad + i\left(\frac{(\omega\otimes c)^T(\omega\otimes a)}{\sqrt{(\omega\otimes a)^T(\omega\otimes a)}} + \frac{(\omega\otimes a)^T(\omega\otimes a)(\omega\otimes d)^T(\omega\otimes b) - (\omega\otimes d)^T(\omega\otimes a)(\omega\otimes b)^T(\omega\otimes a)}{\sqrt{(\omega\otimes a)^T(\omega\otimes a)}R}\right)\Bigg] \\
&\quad \times \Bigg[-\frac{(\omega\otimes b)^T(\omega\otimes a)}{\sqrt{(\omega\otimes a)^T(\omega\otimes a)}} + i\Bigg(\frac{(\omega\otimes a)^T(\omega\otimes a)}{\sqrt{(\omega\otimes a)^T(\omega\otimes a)}} \\
&\quad + \frac{(\omega\otimes a)^T(\omega\otimes a)(\omega\otimes b)^T(\omega\otimes b) - ((\omega\otimes b)^T(\omega\otimes a))^2}{\sqrt{(\omega\otimes a)^T(\omega\otimes a)}R}\Bigg)\Bigg]^{-1} \\
&= [(\omega\otimes a)^T(\omega\otimes a)(\omega\otimes c)^T(\omega\otimes b) - (\omega\otimes c)^T(\omega\otimes a)(\omega\otimes b)^T(\omega\otimes a) - (\omega\otimes d)^T(\omega\otimes a)R \\
&\quad + i[(\omega\otimes c)^T(\omega\otimes a)R + (\omega\otimes a)^T(\omega\otimes a)(\omega\otimes d)^T(\omega\otimes b) - (\omega\otimes d)^T(\omega\otimes a)(\omega\otimes b)^T(\omega\otimes a)]] \\
&\quad \times \{-(\omega\otimes b)^T(\omega\otimes a)R + i[(\omega\otimes a)^T(\omega\otimes a)R + (\omega\otimes a)^T(\omega\otimes a)(\omega\otimes b)^T(\omega\otimes b) \\
&\quad - ((\omega\otimes b)^T(\omega\otimes a))^2]\}^{-1} \\
&= [(\omega\otimes a)^T(\omega\otimes a)(\omega\otimes c)^T(\omega\otimes b) - (\omega\otimes c)^T(\omega\otimes a)(\omega\otimes b)^T(\omega\otimes a) - (\omega\otimes d)^T(\omega\otimes a)R \\
&\quad + i[(\omega\otimes c)^T(\omega\otimes a)R + (\omega\otimes a)^T(\omega\otimes a)(\omega\otimes d)^T(\omega\otimes b) - (\omega\otimes d)^T(\omega\otimes a)(\omega\otimes b)^T(\omega\otimes a)]] \\
&\quad \times \{-(\omega\otimes b)^T(\omega\otimes a)R + iR[(\omega\otimes a)^T(\omega\otimes a) + R]\}^{-1} \\
&= [-(\omega\otimes d)^T(\omega\otimes a) + i(\omega\otimes c)^T(\omega\otimes a) - i[(\omega\otimes a)^T(\omega\otimes a)(-(\omega\otimes d)^T(\omega\otimes b) + i(\omega\otimes c)^T(\omega\otimes b)) \\
&\quad - (-(\omega\otimes d)^T(\omega\otimes a) + i(\omega\otimes c)^T(\omega\otimes a))(\omega\otimes b)^T(\omega\otimes a)]/R] \\
&\quad \times [-(\omega\otimes b)^T(\omega\otimes a) + i(\omega\otimes a)^T(\omega\otimes a) + iR]^{-1}.
\end{aligned}$$

Here, we note that

$$(\omega\otimes a)^T(\omega\otimes a)=\sum_{j=0}^{n-1}\omega_j^2a_j^2=\sum_{j=0}^{n-1}\omega_j^2\left(\frac{z^j+\bar{z}^j}{2}\right)^2=\sum_{j=0}^{n-1}\frac{1}{4}\omega_j^2(z^{2j}+2|z|^{2j}+\bar{z}^{2j}),$$

$$(\omega\otimes b)^T(\omega\otimes a)=\sum_{j=0}^{n-1}\omega_j^2b_ja_j=\sum_{j=0}^{n-1}\omega_j^2\left(\frac{z^j-\bar{z}^j}{2i}\right)\left(\frac{z^j+\bar{z}^j}{2}\right)=\sum_{j=0}^{n-1}-\frac{i}{4}\omega_j^2(z^{2j}-\bar{z}^{2j}),$$

$$(\omega\otimes b)^T(\omega\otimes b)=\sum_{j=0}^{n-1}\omega_j^2b_j^2=\sum_{j=0}^{n-1}\omega_j^2\left(\frac{z^j-\bar{z}^j}{2i}\right)^2=\sum_{j=0}^{n-1}-\frac{1}{4}\omega_j^2(z^{2j}-2|z|^{2j}+\bar{z}^{2j}),$$

$$(\omega\otimes c)^T(\omega\otimes a)=\sum_{j=0}^{n-1}\omega_j^2c_ja_j=\sum_{j=0}^{n-1}j\omega_j^2a_j^2=\sum_{j=0}^{n-1}\frac{j}{4}\omega_j^2(z^{2j}+2|z|^{2j}+\bar{z}^{2j}),$$

$$(\omega\otimes c)^T(\omega\otimes b)=\sum_{j=0}^{n-1}\omega_j^2c_jb_j=\sum_{j=0}^{n-1}j\omega_j^2a_jb_j=\sum_{j=0}^{n-1}j\omega_j^2\left(\frac{z^j+\bar{z}^j}{2}\right)\left(\frac{z^j-\bar{z}^j}{2i}\right)=\sum_{j=0}^{n-1}-\frac{ij}{4}\omega_j^2(z^{2j}-\bar{z}^{2j}),$$

$$(\omega\otimes d)^T(\omega\otimes a)=\sum_{j=0}^{n-1}\omega_j^2d_ja_j=\sum_{j=0}^{n-1}j\omega_j^2b_ja_j=\sum_{j=0}^{n-1}j\omega_j^2\left(\frac{z^j-\bar{z}^j}{2i}\right)\left(\frac{z^j+\bar{z}^j}{2}\right)=\sum_{j=0}^{n-1}-\frac{ij}{4}\omega_j^2(z^{2j}-\bar{z}^{2j}),$$

$$(\omega\otimes d)^T(\omega\otimes b)=\sum_{j=0}^{n-1}\omega_j^2d_jb_j=\sum_{j=0}^{n-1}j\omega_j^2b_j^2=\sum_{j=0}^{n-1}j\omega_j^2\left(\frac{z^j-\bar{z}^j}{2i}\right)^2=\sum_{j=0}^{n-1}-\frac{j}{4}\omega_j^2(z^{2j}-2|z|^{2j}+\bar{z}^{2j}).$$

Let

$$A_{k,n}(z)=\sum_{j=0}^{n-1}j^k\omega_j^2z^{2j}$$

with $k=0,1$, and let

$$B_{k,n}(z)=\sum_{j=0}^{n-1}j^k\omega_j^2|z|^{2j}$$

with $k=0,1,2$. We write

$$(\omega\otimes a)^T(\omega\otimes a)=\frac{1}{4}(A_{0,n}+2B_{0,n}+\bar{A}_{0,n}),$$

$$(\omega\otimes b)^T(\omega\otimes a)=-\frac{i}{4}(A_{0,n}-\bar{A}_{0,n}),\qquad (\omega\otimes b)^T(\omega\otimes b)=-\frac{1}{4}(A_{0,n}-2B_{0,n}+\bar{A}_{0,n}),$$

$$(\omega\otimes c)^T(\omega\otimes a)=\frac{1}{4}(A_{1,n}+2B_{1,n}+\bar{A}_{1,n}),\quad (\omega\otimes c)^T(\omega\otimes b)=-\frac{i}{4}(A_{1,n}+\bar{A}_{1,n}),$$

$$(\omega\otimes d)^T(\omega\otimes a)=-\frac{i}{4}(A_{1,n}-\bar{A}_{1,n}),\qquad (\omega\otimes d)^T(\omega\otimes b)=-\frac{i}{4}(A_{1,n}-2B_{1,n}+\bar{A}_{1,n}).$$

Also, putting

$$R=\frac{1}{2}\sqrt{B_{0,n}^2-|A_{0,n}|^2}$$

and

$$D_{0,n}=2R=\sqrt{B_{0,n}^2-|A_{0,n}|^2},$$

we substitute these formulas into the last expression for F and use

$$\overline{A_{0,n}+B_{0,n}+D_{0,n}}=\bar{A}_{0,n}+B_{0,n}+D_{0,n}$$

to compute that

$$\begin{aligned}
F(z) &= \frac{\dfrac{i}{2}(A_{0,n}+B_{0,n}) + \dfrac{\dfrac{i}{4}(A_{0,n}B_{1,n}+B_{0,n}B_{1,n}-A_{1,n}B_{0,n}-\bar{A}_{0,n}A_{1,n})}{\dfrac{1}{2}\sqrt{B_{0,n}^2-|A_{0,n}|^2}}}{\dfrac{i}{2}(A_{0,n}+B_{0,n}) + \dfrac{i}{2}\sqrt{B_{0,n}^2-|A_{0,n}|^2}} \\
&= \frac{A_{1,n}+B_{1,n}+(A_{0,n}B_{1,n}+B_{0,n}B_{1,n}-A_{1,n}B_{0,n}-\bar{A}_{0,n}A_{1,n})/D_{0,n}}{A_{0,n}+B_{0,n}+D_{0,n}} \\
&= \frac{[A_{1,n}+B_{1,n}+(A_{0,n}B_{1,n}+B_{0,n}B_{1,n}-A_{1,n}B_{0,n}-\bar{A}_{0,n}A_{1,n})/D_{0,n}]\overline{(A_{0,n}+B_{0,n}+D_{0,n})}}{(A_{0,n}+B_{0,n}+D_{0,n})\overline{(A_{0,n}+B_{0,n}+D_{0,n})}} \\
&= \frac{(A_{1,n}D_{0,n}+B_{1,n}D_{0,n}+A_{0,n}B_{1,n}+B_{0,n}B_{1,n}-A_{1,n}B_{0,n}-\bar{A}_{0,n}A_{1,n})(\bar{A}_{0,n}+B_{0,n}+D_{0,n})}{D_{0,n}(A_{0,n}+B_{0,n}+D_{0,n})(\bar{A}_{0,n}+B_{0,n}+D_{0,n})}.
\end{aligned}$$

The denominator is made real by multiplying and dividing by its complex conjugate. We have

$$\begin{aligned}
&(A_{0,n}+B_{0,n}+D_{0,n})(\bar{A}_{0,n}+B_{0,n}+D_{0,n}) \\
&\quad = A_{0,n}\bar{A}_{0,n}+A_{0,n}B_{0,n}+A_{0,n}D_{0,n}+\bar{A}_{0,n}B_{0,n}+B_{0,n}^2+2B_{0,n}D_{0,n}+\bar{A}_{0,n}D_{0,n}+D_{0,n}^2 \\
&\quad = |A_{0,n}|^2+A_{0,n}B_{0,n}+A_{0,n}D_{0,n}+\bar{A}_{0,n}B_{0,n}+B_{0,n}^2+2B_{0,n}D_{0,n}+\bar{A}_{0,n}D_{0,n}+(B_{0,n}^2-|A_{0,n}^2|) \\
&\quad = A_{0,n}B_{0,n}+2B_{0,n}^2+\bar{A}_{0,n}B_{0,n}+A_{0,n}D_{0,n}+2B_{0,n}D_{0,n}+\bar{A}_{0,n}D_{0,n} \\
&\quad = B_{0,n}(A_{0,n}+2B_{0,n}+\bar{A}_{0,n})+D_{0,n}(A_{0,n}+2B_{0,n}+\bar{A}_{0,n}) \\
&\quad = (B_{0,n}+D_{0,n})(A_{0,n}+2B_{0,n}+\bar{A}_{0,n})
\end{aligned}$$

and

$$\begin{aligned}
&(A_{1,n}D_{0,n}+B_{1,n}D_{0,n}+A_{0,n}B_{1,n}+B_{0,n}B_{1,n}-A_{1,n}B_{0,n}-\bar{A}_{0,n}A_{1,n})(\bar{A}_{0,n}+B_{0,n}+D_{0,n}) \\
&\quad = A_{1,n}(B_{0,n}^2-|A_{0,n}|^2)+\bar{A}_{0,n}B_{1,n}D_{0,n}+2B_{0,n}B_{1,n}D_{0,n}+B_{1,n}(B_{0,n}^2-\bar{A_{0,n}}^2)+\bar{A}_{0,n}A_{0,n}B_{1,n} \\
&\qquad + A_{0,n}B_{0,n}B_{1,n}+A_{0,n}B_{1,n}D_{0,n}+\bar{A}_{0,n}B_{0,n}B_{1,n}+B_{0,n}^2B_{1,n}-2\bar{A}_{0,n}A_{1,n}B_{0,n}-A_{1,n}B_{0,n}^2-\bar{A}_{0,n}^2A_{1,n} \\
&\quad = -\bar{A}_{0,n}A_{0,n}A_{1,n}+\bar{A}_{0,n}B_{1,n}D_{0,n}+2B_{0,n}B_{1,n}D_{0,n}+2B_{0,n}^2B_{1,n}+A_{0,n}B_{0,n}B_{1,n}+A_{0,n}B_{1,n}D_{0,n} \\
&\qquad + \bar{A}_{0,n}B_{0,n}B_{1,n}-2\bar{A}_{0,n}A_{1,n}B_{0,n}-\bar{A}_{0,n}^2A_{1,n} \\
&\quad = -(A_{0,n}+2B_{0,n}+\bar{A}_{0,n})(\bar{A}_{0,n}A_{1,n}-B_{1,n}(B_{0,n}+D_{0,n})).
\end{aligned}$$

Hence,

$$\begin{aligned}
F(z) &= \frac{B_{1,n}D_{0,n}+B_{0,n}B_{1,n}-\bar{A}_{0,n}A_{1,n}}{D_{0,n}(B_{0,n}+D_{0,n})} \\
&= \frac{B_{1,n}D_{0,n}+B_{0,n}B_{1,n}-\bar{A}_{0,n}A_{1,n}}{B_{0,n}D_{0,n}+B_{0,n}^2-\bar{A}_{0,n}A_{0,n}},
\end{aligned}$$

and it remains to find the expectation $\mathbb{E}[\nu_n(\Omega)]$.

We use Stokes' theorem in the plane, otherwise called Green's theorem, which is that

$$\int_{\partial D}(u(x,y)\,dx+v(x,y)\,dy) = \iint_D \left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)dx\,dy,$$

for a simply connected region D with a sufficiently smooth boundary ∂D, where $u(x,y)$ and $v(x,y)$ are one-valued functions having continuous first partial derivatives in ∂D. The complex analogue of Stokes' theorem is that

$$\int_{\partial D}(U\,dz+V\,d\bar{z}) = \iint_D \left(\frac{\partial V}{\partial z}-\frac{\partial U}{\partial \bar{z}}\right)dz\,d\bar{z}.$$

(See Buck [2], Theorem 11 and the remark following it, page 209.) We now write $U(z,\bar{z})$ and $V(z,\bar{z})$ to emphasize the fact that U and V each depend on both z and $\bar{z}$. Applying the formula above with

$U(z,\bar z)=(2\pi i z)^{-1}F(z,\bar z)$, $V(z,\bar z)=0$, and $dz\,d\bar z=-2i\,dx\,dy$ to

$$\mathbb{E}[\nu_n(\Omega)]=\frac{1}{2\pi i}\int_{\partial\Omega}\frac{1}{z}F(z,\bar z)\,dz,$$

we obtain

$$\begin{aligned}\mathbb{E}[\nu_n(\Omega)]&=-\iint_{\Omega}\frac{\partial}{\partial\bar z}\left(\frac{1}{2\pi i z}F(z,\bar z)\right)dz\,d\bar z\\ &=\frac{1}{\pi}\iint_{\Omega}\frac{1}{z}\frac{\partial}{\partial\bar z}F(z,\bar z)\,dx\,dy.\end{aligned}$$

Letting the symbol † denote the derivative with respect to $\bar z$ and using the quotient rule, we obtain

$$\begin{aligned}\frac{\partial}{\partial\bar z}F(z,\bar z)&=\left(\frac{B_{1,n}D_{0,n}+B_{0,n}B_{1,n}-\bar A_{0,n}A_{1,n}}{B_{0,n}D_{0,n}+B_{0,n}^2-\bar A_{0,n}A_{0,n}}\right)^{\dagger}\\ &=[(B_{0,n}D_{0,n}+B_{0,n}^2-\bar A_{0,n}A_{0,n})(B_{1,n}D_{0,n}+B_{0,n}B_{1,n}-\bar A_{0,n}A_{1,n})^{\dagger}-(B_{1,n}D_{0,n}\\ &\quad+B_{0,n}B_{1,n}-\bar A_{0,n}A_{1,n})(B_{0,n}D_{0,n}+B_{0,n}^2-|A_{0,n}|^2)^{\dagger}]/(B_{0,n}D_{0,n}+B_{0,n}^2-|A_{0,n}|^2)^2\\ &=[(B_{0,n}D_{0,n}+B_{0,n}^2-|A_{0,n}|^2)(B_{1,n}^{\dagger}D_{0,n}+B_{1,n}D_{0,n}^{\dagger}+B_{0,n}^{\dagger}B_{1,n}+B_{0,n}B_{1,n}^{\dagger}\\ &\quad-\bar A_{0,n}^{\dagger}A_{1,n}-\bar A_{0,n}A_{1,n}^{\dagger})-(B_{1,n}D_{0,n}+B_{0,n}B_{1,n}-\bar A_{0,n}A_{1,n})(B_{0,n}^{\dagger}D_{0,n}+B_{0,n}D_{0,n}^{\dagger}\\ &\quad+2B_{0,n}B_{0,n}^{\dagger}-\bar A_{0,n}^{\dagger}A_{0,n}-\bar A_{0,n}A_{0,n}^{\dagger})]/(B_{0,n}D_{0,n}+B_{0,n}^2-|A_{0,n}|^2)^2\\ &=[(B_{0,n}D_{0,n}+B_{0,n}^2-|A_{0,n}|^2)(B_{1,n}^{\dagger}D_{0,n}+B_{1,n}D_{0,n}^{\dagger}+B_{0,n}^{\dagger}B_{1,n}+B_{0,n}B_{1,n}^{\dagger}-\bar A_{0,n}^{\dagger}A_{1,n})\\ &\quad-(B_{1,n}D_{0,n}+B_{0,n}B_{1,n}-\bar A_{0,n}A_{1,n})(B_{0,n}^{\dagger}D_{0,n}+B_{,n}D_{0,n}^{\dagger}+2B_{0,n}B_{0,n}^{\dagger}-\bar A_{0,n}^{\dagger}A_{0,n})]\\ &\quad/(B_{0,n}D_{0,n}+B_{0,n}^2-|A_{0,n}|^2)^2,\end{aligned}$$

since $A_{0,n}^{\dagger}=0$ and $A_{1,n}^{\dagger}=0$. We remove the derivatives $\bar A_{0,n}^{\dagger}, B_{0,n}^{\dagger}, B_{1,n}^{\dagger}$, and $D_{0,n}^{\dagger}$ from the last expression and note that

$$\begin{aligned}\bar A_{0,n}&=\sum_{j=0}^{n-1}\omega_j^2\bar z^{2j}, & \bar A_{0,n}^{\dagger}&=2\bar z^{-1}\sum_{j=0}^{n-1}j\omega_j^2\bar z^{2j}=2\bar z^{-1}\bar A_{1,n},\\ B_{0,n}&=\sum_{j=0}^{n-1}\omega_j^2|z|^{2j}=\sum_{j=0}^{n-1}\omega_j^2 z^j\bar z^j, & B_{0,n}^{\dagger}&=\bar z^{-1}\sum_{j=0}^{n-1}j\omega_j^2(z\bar z)^j=\bar z^{-1}\sum_{j=0}^{n-1}j\omega_j^2|z|^{2j}=\bar z^{-1}B_{1,n},\\ B_{1,n}&=\sum_{j=0}^{n-1}j\omega_j^2|z|^{2j}=\sum_{j=0}^{n-1}j\omega_j^2 z^j\bar z^j, & B_{1,n}^{\dagger}&=\bar z^{-1}\sum_{j=0}^{n-1}j^2\omega_j^2(z\bar z)^j=\bar z^{-1}\sum_{j=0}^{n-1}j^2\omega_j^2|z|^{2j}=\bar z^{-1}B_{2,n}.\end{aligned}$$

Applying the chain rule to $D_{0,n}=\sqrt{B_{0,n}^2-|A_{0,n}|^2}$ and using the expressions for $A_{0,n}^{\dagger}, \bar A_{0,n}^{\dagger}$, and $B_{0,n}^{\dagger}$, we obtain

$$\begin{aligned}D_{0,n}^{\dagger}&=\frac{(B_{0,n}^2-|A_{0,n}|^2)^{\dagger}}{2\sqrt{B_{0,n}^2-|A_{0,n}|^2}}\\ &=\frac{2B_{0,n}B_{0,n}^{\dagger}-\bar A_{0,n}^{\dagger}A_{0,n}-\bar A_{0,n}A_{0,n}^{\dagger}}{2D_{0,n}}\\ &=\frac{2B_{0,n}(\bar z^{-1}B_{1,n})-2\bar z^{-1}A_{0,n}\bar A_{1,n}}{2D_{0,n}}\\ &=\frac{B_{0,n}B_{1,n}-A_{0,n}\bar A_{1,n}}{\bar z D_{0,n}}.\end{aligned}$$

Substituting these formulas for the derivatives into the expression given above for $\partial F/\partial \bar{z}$, we find that

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}} F(z,\bar{z}) &= \Bigg\{(B_{0,n}D_{0,n} + B_{0,n}^2 - \bar{A}_{0,n}A_{0,n})\Bigg[\frac{B_{2,n}D_{0,n}}{\bar{z}} + B_{1,n}\left(\frac{B_{0,n}B_{1,n} - A_{0,n}\bar{A}_{1,n}}{\bar{z}D_{0,n}}\right) + \frac{B_{1,n}^2}{\bar{z}} + \frac{B_{0,n}B_{2,n}}{\bar{z}} \\
&\quad - \frac{2\bar{A}_{1,n}A_{1,n}}{\bar{z}}\Bigg] - (B_{1,n}D_{0,n} + B_{0,n}B_{1,n} - \bar{A}_{0,n}A_{1,n})\Bigg[\frac{B_{1,n}D_{0,n}}{\bar{z}} + B_{0,n}\left(\frac{B_{0,n}B_{1,n} - A_{0,n}\bar{A}_{1,n}}{\bar{z}D_{,n}}\right) \\
&\quad + \frac{2B_{0,n}B_{1,n}}{\bar{z}} - \frac{2A_{0,n}\bar{A}_{1,n}}{\bar{z}}\Bigg]\Bigg\}/(B_{0,n}D_{0,n} + B_{0,n}^2 - \bar{A}_{0,n}A_{0,n})^2 \\
&= (B_{0,n}D_{0,n} + B_{0,n}^2 - \bar{A}_{0,n}A_{0,n})(B_{2,n}D_{0,n}^2 + B_{0,n}B_{1,n}^2 - A_{0,n}\bar{A}_{1,n}B_{1,n} + B_{1,n}^2D_{0,n} + B_{0,n}B_{2,n}D_{0,n} \\
&\quad - 2\bar{A}_{1,n}A_{1,n}D_{0,n}) - (B_{1,n}D_{0,n} + B_{0,n}B_{1,n} - \bar{A}_{0,n}A_{1,n})(B_{1,n}D_{0,n}^2 + B_{0,n}^2B_{1,n} - A_{0,n}\bar{A}_{1,n}B_{0,n} \\
&\quad + 2B_{0,n}B_{1,n}D_{0,n} - 2A_{0,n}\bar{A}_{1,n}D_{0,n})/\bar{z}D_{0,n}(B_{0,n}^2D_{0,n}^2 + 2B_{0,n}^3D_{0,n} - 2\bar{A}_{0,n}A_{0,n}B_{0,n}D_{0,n} + B_{0,n}^4 \\
&\quad - 2\bar{A}_{0,n}A_{0,n}B_{0,n}^{2,n} + \bar{A}_{0,n}^2A_{0,n}^2) \\
&= (B_{0,n}B_{2,n}D_{0,n}^3 - B_{0,n}^2B_{1,n}^2D_{0,n} + 2A_{0,n}\bar{A}_{1,n}B_{0,n}B_{1,n}D_{0,n} - 2B_{0,n}B_{1,n}^2D_{0,n}^2 + 2B_{0,n}^2B_{2,n}D_{0,n}^2 \\
&\quad - 2\bar{A}_{1,n}A_{1,n}B_{0,n}D_{0,n}^2 + B_{0,n}^3B_{2,n}D_{0,n} - 2\bar{A}_{1,n}A_{1,n}B_{0,n}^2D_{0,n} - \bar{A}_{0,n}A_{0,n}B_{2,n}D_{0,n}^2 \\
&\quad - \bar{A}_{0,n}A_{0,n}B_{0,n}B_{1,n}^2 + \bar{A}_{0,n}A_{0,n}^2\bar{A}_{1,n}B_{1,n} - \bar{A}_{0,n}A_{0,n}B_{1,n}^2D_{0,n} - \bar{A}_{0,n}A_{0,n}B_{0,n}B_{2,n}D_{0,n} \\
&\quad - B_{1,n}^2D_{0,n}^3 + 2A_{0,n}\bar{A}_{1,n}B_{1,n}D_{0,n}^2 + \bar{A}_{0,n}A_{1,n}B_{1,n}D_{0,n}^2 + \bar{A}_{0,n}A_{1,n}B_{0,n}^2B_{1,n} \\
&\quad - \bar{A}_{0,n}A_{0,n}\bar{A}_{1,n}A_{1,n}B_{0,n} + 2\bar{A}_{0,n}A_{1,n}B_{0,n}B_{1,n}D_{0,n})/\bar{z}D_{0,n}(B_{0,n}^2D_{0,n}^2 + 2B_{0,n}^3D_{0,n} \\
&\quad - 2\bar{A}_{0,n}A_{0,n}B_{0,n}D_{0,n} + B_{0,n}^4 - 2\bar{A}_{0,n}A_{0,n}B_{0,n}^2 + \bar{A}_{0,n}^2A_{0,n}^2).
\end{aligned}$$

From this, we obtain the intensity function h_n given by

$$\frac{1}{\pi z}\left(\frac{\partial}{\partial \bar{z}} F(z,\bar{z})\right) = \frac{B_{2,n}D_{0,n}^2 - B_{0,n}(B_{1,n}^2 + |A_{1,n}|^2) + B_{1,n}(A_{0,n}\bar{A}_{1,n} + \bar{A}_{0,n}A_{1,n})}{\pi |z|^2 D_{0,n}^3}.$$

Next, we consider a point x on the real axis and analyze the limiting behavior of F near the real axis. Here, we have $A_{k,n} = B_{k,n}$ for $k = 0, 1$. Thus $R = 0$, and hence $D_{0,n} = 0$. That is to say, F is indeterminate as z approaches x. Hence,

$$F(z) = \frac{B_{1,n} + \dfrac{B_{0,n}B_{1,n} - \bar{A}_{0,n}A_{1,n}}{D_{0,n}}}{B_{0,n} + D_{0,n}},$$

with the ratio in the numerator being indeterminate. We let $z = re^{i\theta}$ and examine the numerator and denominator of the ratio $(B_{0,n}B_{1,n} - \bar{A}_{0,n}A_{1,n})/D_{0,n}$ when θ is small. We have

$$\begin{aligned}
B_{0,n}B_{1,n} - \bar{A}_{0,n}A_{1,n} &= \sum_{j=0}^{n-1} \omega_j^2 |z|^{2j} \sum_{k=0}^{n-1} k\omega_k^2 |z|^{2k} - \sum_{j=0}^{n-1} \omega_j^2 \bar{z}^{2j} \sum_{k=0}^{n-1} k\omega_k^2 z^{2k} \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2 k\omega_k^2 r^{2(j+k)} - \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2 k\omega_k^2 r^{2(j+k)} e^{2(k-j)\theta i} \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2 k\omega_k^2 r^{2(j+k)} (1 - e^{2(k-j)\theta i}) \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2 k\omega_k^2 r^{2(j+k)} 2(j-k)\theta i + o(\theta) \\
&= 2\theta i\left(\sum_{j=0}^{n-1} j\omega_j^2 r^{2j} \sum_{k=0}^{n-1} k\omega_k^2 r^{2k} - \sum_{j=0}^{n-1} \omega_j^2 r^{2j} \sum_{k=0}^{n-1} k^2\omega_k^2 r^{2k}\right) + o(\theta) \\
&= 2\theta i(B_{1,n}^2 - B_{0,n}B_{2,n}) + o(\theta),
\end{aligned}$$

and

$$\begin{aligned}
D_{0,n}^2 &= B_{0,n}^2 - |A_{0,n}|^2 = B_{0,n}^2 - \bar{A}_{0,n}A_{0,n} \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2\omega_k^2 r^{2(j+k)} - \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2\omega_k^2 r^{2(j+k)} e^{2(k-j)\theta i} \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2\omega_k^2 r^{2(j+k)} (1 - e^{2(k-j)\theta i}) \\
&= \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2\omega_k^2 r^{2(j+k)} \left[-\frac{(2(k-j)\theta i)^2}{2!} \right] + o(\theta^2) \\
&= -\sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2\omega_k^2 r^{2(j+k)} 2(k-j)^2\theta^2 i^2 + o(\theta^2) \\
&= 2\theta^2 \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2\omega_k^2 r^{2(j+k)} k^2 - 4\theta^2 \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2\omega_k^2 r^{2(j+k)} jk + 2\theta^2 \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \omega_j^2\omega_k^2 r^{2(j+k)} j^2 \\
&= 2\theta^2 \sum_{j=0}^{n-1} \omega_j^2 r^{2j} \sum_{k=0}^{n-1} k^2\omega_k^2 r^{2k} - 4\theta^2 \sum_{j=0}^{n-1} j\omega_j^2 r^{2j} \sum_{k=0}^{n-1} k\omega_k^2 r^{2k} + 2\theta^2 \sum_{j=0}^{n-1} j^2\omega_j^2 r^{2j} \sum_{k=0}^{n-1} \omega_k^2 r^{2k} + o(\theta^2) \\
&= 4\theta^2 \left(\sum_{j=0}^{n-1} \omega_j^2 r^{2j} \sum_{k=0}^{n-1} k^2\omega_k^2 r^{2k} - \sum_{j=0}^{n-1} j\omega_j^2 r^{2j} \sum_{k=0}^{n-1} k\omega_k^2 r^{2k} \right) + o(\theta^2) \\
&= 4\theta^2 (B_{0,n}B_{2,n} - B_{1,n}^2) + o(\theta^2).
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{B_{0,n}B_{1,n} - \bar{A}_{0,n}A_{1,n}}{D_{0,n}} &= \frac{2\theta i(B_{1,n}^2 - B_{0,n}B_{2,n}) + o(\theta)}{\sqrt{4\theta^2(B_{0,n}B_{2,n} - B_{1,n}^2) + o(\theta^2)}} \\
&= \operatorname{sgn}(\theta) i \frac{B_{1,n}^2 - B_{0,n}B_{2,n} + o(\theta)}{\sqrt{B_{0,n}B_{2,n} - B_{1,n}^2 + o(\theta^2)}} \\
&= -\operatorname{sgn}(\theta) i \sqrt{B_{0,n}B_{2,n} - B_{1,n}^2} + o(\theta),
\end{aligned}$$

where sgn is the signum function. Hence,

$$F(z) = \frac{B_{1,n} - \operatorname{sgn}(\theta) i \sqrt{B_{0,n}B_{2,n} - B_{1,n}^2} + o(\theta)}{B_{0,n} + D_{0,n}},$$

from which we deduce that

$$\lim_{\substack{z \to x \\ \operatorname{Im}(z) > 0}} F(z) = \frac{B_{1,n} - i\sqrt{B_{0,n}B_{2,n} - B_{1,n}^2}}{B_{0,n}}$$

and

$$\lim_{\substack{z \to x \\ \operatorname{Im}(z) < 0}} F(z) = \frac{B_1 + i\sqrt{B_{0,n}B_{2,n} - B_{1,n}^2}}{B_{0,n}}.$$

We consider a polar rectangle that covers a portion of the real axis. Let Ω be the angular interval $(-\theta, \theta)$ crossed with a radial interval (r_0, r_1), so that $\Omega \cap \mathbb{R} = (r_0, r_1)$. Let, further, $\nu_n((r_0, r_1))$ denote the number of zeros in the interval (r_0, r_1) of the real axis. Without loss of generality, we may assume that the polar rectangle does not intersect the negative part of the real axis. Then, using

$$\mathbb{E}[\nu_n(\Omega)] = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{1}{z} F(z)\, dz,$$

where $z = re^{i\theta}$, and letting $\theta \to 0$, we find that

$$\mathbb{E}[\nu_n((r_0, r_1))] = \frac{1}{2\pi i} \int_{r_0}^{r_1} \frac{F(r^-) - F(r^+)}{r} \, dr,$$

where

$$F(r^-) = \lim_{\substack{z \to r \\ \operatorname{Im}(z) < 0}} F(z) = \frac{B_{1,n} - i\sqrt{B_{0,n}B_{2,n} - B_{1,n}^2}}{B_{0,n}}$$

and

$$F(r^+) = \lim_{\substack{z \to r \\ \operatorname{Im}(z) > 0}} F(z) = \frac{B_{1,n} + i\sqrt{B_{0,n}B_{2,n} - B_{1,n}^2}}{B_{0,n}}.$$

Hence, we obtain the intensity function g_n, which is defined by

$$g_n(r) = \frac{F(r^-) - F(r^+)}{2\pi i r} = \frac{\sqrt{B_{0,n}B_{2,n} - B_{1,n}^2}}{\pi r B_{0,n}}.$$

This completes the proof of Theorem 1.

We refrain from providing the details of the proof of Theorem 2, since the reasoning is similar to that in the proof of Theorem 1. However, we note here that the random variables ξ_1, ξ_2, ξ_3, and ξ_4 take the form

$$\xi_1 = \sum_{j=0}^{n-1} \omega_j(\alpha_j a_j - \beta_j b_j), \quad \xi_2 = \sum_{j=0}^{n-1} \omega_j(\alpha_j b_j - \beta_j a_j),$$
$$\xi_3 = \sum_{j=0}^{n-1} \omega_j(\alpha_j c_j - \beta_j d_j), \quad \xi_4 = \sum_{j=0}^{n-1} \omega_j(\alpha_j d_j + \beta_j c_j),$$

and that the covariance matrix

$$\operatorname{Cov}(\xi) = \begin{bmatrix} \sum_{j=0}^{n-1} \omega_j^2 |z|^{2j} & 0 & \sum_{j=0}^{n-1} \omega_j^2 j |z|^{2j} & 0 \\ 0 & \sum_{j=0}^{n-1} \omega_j^2 |z|^{2j} & 0 & \sum_{j=0}^{n-1} \omega_j^2 j |z|^{2j} \\ \sum_{j=0}^{n-1} \omega_j^2 j |z|^{2j} & 0 & \sum_{j=0}^{n-1} \omega_j^2 j^2 |z|^{2j} & 0 \\ 0 & \sum_{j=0}^{n-1} \omega_j^2 j |z|^{2j} & 0 & \sum_{j=0}^{n-1} \omega_j^2 j^2 |z|^{2j} \end{bmatrix}$$

plays a crucial role in the proof.

3. Proof of Corollary 1

In this section, we take up the proof of Corollary 1. We refrain from giving the details of the proof of Corollary 2, since the reasoning is similar to that in the proof of Corollary 1.

To prove part (i), we observe that

$$\frac{1}{j+1}\binom{n-1}{j} = \frac{n!}{n(j+1)!(n-j-1)!} = \frac{1}{n}\binom{n}{j+1}.$$

By the binomial theorem, we see that $B_{0,n}$ is equal to

$$\frac{1}{n}\sum_{j=0}^{n-1}\binom{n}{j+1}|z|^{2j} = \frac{1}{n|z|^2}\sum_{j=1}^{n}\binom{n}{j}|z|^{2j} = \frac{(|z|^2+1)^n - 1}{n|z|^2}.$$

By repeated differentiation with respect to $|z|^2$ and the binomial theorem, we find that $B_{1,n}$ is equal to

$$\frac{1}{n}\sum_{j=0}^{n-1}\binom{n}{j+1}j|z|^{2j} = \frac{|z|^2}{n}\frac{d}{d|z|^2}\left(\sum_{j=0}^{n-1}\binom{n}{j+1}|z|^{2j}\right) = \frac{(|z|^2+1)^{n-1}((n-1)|z|^2-1)+1}{n|z|^2}$$

and that $B_{2,n}$ is equal to

$$\frac{1}{n}\sum_{j=0}^{n-1}\binom{n}{j+1}j^2|z|^{2j} = \frac{|z|^2}{n}\frac{d}{d|z|^2}\left(\sum_{j=0}^{n-1}\binom{n}{j+1}j|z|^{2j}\right) = \frac{(|z|^2+1)^{n-2}((n-1)^2|z|^4-(n-2)|z|^2+1)}{n|z|^2}.$$

If we assume $|z|>0$, then for n sufficiently large

$$B_{0,n}\sim\frac{(|z|^2+1)^n}{n|z|^2},\quad B_{1,n}\sim(|z|^2+1)^{n-1},\quad B_{2,n}\sim n|z|^2(|z|^2+1)^{n-2}.$$

By the same argument as above, we get that for n sufficiently large

$$A_{0,n}\sim\frac{(z^2+1)^n}{nz^2},\quad A_{1,n}\sim(z^2+1)^{n-1}.$$

We obtain from Theorem 1 the required intensity functions h_n and g_n valid for $|z|>0$ and n sufficiently large.

To prove part (ii), we observe that

$$B_{0,n} = \sum_{j=0}^{n-1} j|z|^{2j} = |z|^2\left(\frac{d}{d|z|^2}\sum_{j=0}^{n-1}|z|^{2j}\right) = \frac{-n|z|^{2n}(1-|z|^2)+|z|^2(1-|z|^{2n})}{(1-|z|^2)^2},$$

$$B_{1,n} = \sum_{j=0}^{n-1} j^2|z|^{2j} = |z|^2\left(\frac{d}{d|z|^2}\sum_{j=0}^{n-1}j|z|^{2j}\right) = \frac{-n^2|z|^{2n}(1-|z|^2)^2+|z|^2(1+|z|^2)(1-|z|^{2n})}{(1-|z|^2)^3},$$

and

$$\begin{aligned}B_{2,n} &= \sum_{j=0}^{n-1} j^3|z|^{2j} = |z|^2\left(\frac{d}{d|z|^2}\sum_{j=0}^{n-1}j^j|z|^{2j}\right)\\ &= (-n^3|z|^{2n}(1-|z|^2)^3-3n^2|z|^{2n+2}(1-|z|^2)^2-3n|z|^{2n+2}(1+|z|^2)(1-|z|^2)\\ &\quad+|z|^2(1-|z|^2n)(|z|^4+4|z|^2+1))\times\frac{1}{(1-|z|^2)^4}.\end{aligned}$$

We have similar expressions for $A_{0,n}$ and $A_{1,n}$, but with $|z|$ replaced by z.

For brevity's sake, we let

$$p=1+|z|^2,\quad q=1-|z|^2,\quad t=1+z^2,\quad u=1-z^2,\quad v=|z|^4+4|z|^2+1.$$

We obtain the intensity function h_n. First, we assume that $0<|z|<1$. We have

$$\lim_{n\to\infty}B_{0,n}=\frac{|z|^2}{(1-|z|^2)^2}=\frac{|z|^2}{q^2},$$

$$\lim_{n\to\infty}B_{1,n}=\frac{|z|^2(1+|z|^2)}{(1-|z|^2)^3}=\frac{p|z|^2}{q^3},$$

and

$$\lim_{n\to\infty}B_{2,n}=\frac{|z|^2(|z|^4+4|z|^2+1)}{(1-|z|^2)^4}=\frac{tz^2}{u^3}.$$

We obtain from Theorem 1 the intensity function

$$\begin{aligned}h_n(z) &\sim \frac{1}{\pi|z|^2}\left(\frac{1}{q^4}-\frac{1}{|u|^4}\right)^{-3/2}\left[\frac{v}{q^4}\left(\frac{1}{q^4}-\frac{1}{|u|^4}\right)-\frac{1}{q^2}\left(\frac{p^2}{q^6}+\frac{|t|^2}{|u|^6}\right)+\frac{p}{q^3}\left(\frac{\bar{t}}{u^2\bar{u}^3}+\frac{t}{\bar{u}^2u^3}\right)\right]\\ &= \frac{1}{\pi|z|^2q^2(|u|^4-q^4)^{3/2}}[v(|u|^6-q^4|u|^2)-(p^2|u|^6+q^6|t|^2)+pq^5(\bar{t}u+t\bar{u})].\end{aligned}$$

On noting that $\bar{t}u + t\bar{u} = 2pq$, we obtain the required intensity function h_n valid for $0 < |z| < 1$. Second, we assume that $|z| > 1$. For n sufficiently large, we have

$$\begin{aligned} A_{0,n} &\sim -\frac{z^{2n}(nu + z^2)}{u^2}, \\ A_{1,n} &\sim -\frac{z^{2n}(n^2u^2 + 2nz^2u + z^2t)}{u^3}, \\ B_{0,n} &\sim -\frac{|z|^{2n}(nq + |z|^2)}{q^2}, \\ B_{1,n} &\sim -\frac{|z|^{2n}(n^2q^2 + 2n|z|^2q + |z|^2p)}{q^3}, \\ B_{2,n} &\sim -\frac{|z|^{2n}(n^3q^3 + 3n^2|z|^2q^2 + 3n|z|^2pq)}{q^4}. \end{aligned}$$

Inserting these into Theorem 1 and simplifying we obtain the intensity function h_n valid for $|z| > 1$.

Next, we obtain the intensity function g_n. If $0 < |z| < 1$, using

$$p = 1 + x^2, \quad q = 1 - x^2, \quad v = x^{24} + 4x^2 + 1,$$

we compute that

$$\lim_{n\to\infty} g_n = \frac{\sqrt{v - p^2}}{\pi|x|q} = \frac{\sqrt{2}}{\pi(1 - |x|^2)}.$$

If $|z| > 1$, we have

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} \frac{\sqrt{n^2x^2q^2(p - x^2) + o(n^2)}}{n\pi|x|q^2 + o(n)} = \frac{1}{\pi|q|} = \frac{1}{\pi(x^2 - 1)}.$$

To prove part (iii), we observe that

$$\begin{aligned} A_{0,n} &= \sum_{j=0}^{n-1} \frac{z^{2j}}{j!} \sim e^{z^2}, \\ A_{1,n} &= \sum_{j=1}^{n-1} \frac{z^{2j}}{(j-1)!} = z^2 \sum_{j=0}^{n-2} \frac{z^{2j}}{j!} \sim z^2 e^{z^2}, \\ B_{0,n} &= \sum_{j=0}^{n-1} \frac{|z|^{2j}}{j!} \sim e^{|z|^2}, \\ B_{1,n} &= \sum_{j=1}^{n-1} \frac{|z|^{2j}}{(j-1)!} = |z|^2 \sum_{j=0}^{n-2} \frac{|z|^{2j}}{j!} \sim |z|^2 e^{|z|^2}, \\ B_{2,n} &= \sum_{j=1}^{n-1} \frac{j|z|^{2j}}{(j-1)!} = |z|^2 \frac{d}{d|z|^2} \left(\sum_{j=1}^{n-1} \frac{|z|^{2j}}{(j-1)!} \right) \sim |z|^2 e^{|z|^2}(|z|^2 + 1). \end{aligned}$$

From Theorem 1 we obtain the intensity functions

$$h_n(z) \sim \frac{e^{|z|^2}}{\pi} \left(\frac{e^{2|z|^2} - e^{z^2+\bar{z}^2} + (z - \bar{z})^2 e^{z^2+\bar{z}^2}}{(e^{2|z|^2} - e^{z^2+\bar{z}^2})^{3/2}} \right)$$

and

$$g_n(x) = \frac{\sqrt{x^2e^{2x^2}(x^2 + 1) - x^4e^{2x^2}}}{\pi|x|e^{x^2}} = \frac{1}{\pi}.$$

On passing to the limit, these yield the required expressions for h_n and g_n. This finishes the proof of Corollary 1.

4. Numerical computation

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Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, Tennessee 37240, United States of America

E-mail address: `ethan.l.addison@vanderbilt.edu`

Department of Mathematics, University of Tennessee at Chattanooga, 415 EMCS Building (Mail Stop 6956), 615 McCallie Avenue, Chattanooga, Tennessee 37403, United States of America

E-mail address: `andrew-ledoan@utc.edu`

Department of Mathematics, Southwest Baptist University, 1600 University Avenue, Bolivar, Missouri 65613, United States of America

E-mail address: `s654944@sbuniv.edu`

Mathematics Department, Tennessee Technological University, 110 University Drive (Box 5054), Cookeville, TN 38505, United States of America

E-mail address: `rwvradenbu43@students.tntech.edu`