Ethan L Addison

621 Travers Circle, Apt C
Mishawaka, IN 46545
(423) 328-4452
eaddison@nd.edu

Research Statement

Canonical Kähler Metrics

A persistent theme in complex geometry over the decades is to seek extensions of properties which are easily shown to be true for Riemann surfaces as a result of happy accidents related their one dimensional nature, such as the classical Uniformization Theorem. One such option was introduced by Calabi in [5] as minimizers of the L^2-norm of the scalar curvatures of all metrics in a given Kähler class $[\omega_0]$ on a compact Kähler manifold (M^n, ω_0), i.e. extrema of the functional

$$\omega \mapsto \int_M \mathrm{Scal}(\omega)^2 \omega^n$$

defined for any $\omega \in [\omega_0]$. These extremal metrics also minimize the L^2-norms of the Ricci and Riemann curvatures and satisfy the Euler-Lagrange for the functional which demands the vector field $\mathrm{grad}_\omega \mathrm{Scal}(\omega)$ be holomorphic, highlighting an intimacy between the biholomorphisms and this flavor of metric. Immediately from this characterization, we see that constant scalar curvature metrics and consequentially Kähler-Einsten metrics are extremal and provide some of the most natural and fundamental examples of extremal manifolds.

My work to date has concerned the behavior of non-compact manifolds admitting Kähler metrics which satisfy the sixth-order differential equation governing the holomorphy of the gradient of the scalar curvature. Despite various obstructions to universal existence of extremal metrics on compact manifolds as in [4], for which there are even more in the cscK and KE cases, these metrics remain top contenders for canonical geometries on complex manifolds, particularly in the context of algebraic geometry. Projective varieties can be equipped with line bundles defining their embedding to projective space; by Kodaira [13] the first Chern class of such a line bundle contains Kähler metrics wherein one could possibly associate a canonical metric with the algebraic properties of the embedding. Such an association appears quite delicate and is at the heart of the Yau-Tian-Donaldson Conjecture:

Conjecture 1. *A polarized manifold (M, L) admits an extremal metric in the class $c_1(L)$ iff the polarization is suitably "stable".*

What precisely this sort of stability should be has the subject of generative interest for years, and has been resolved in the past decade for the KE case by Chen-Donaldson-Sun [7].

Generalizing Metrics of Poincaré Type

Even on manifolds for which the variational problem is not well-posed, as in certain non-compact Kähler manifolds, we still define metrics whose scalar curvatures are holomorphy potentials to be extremal. The bulk of my research has concerned complete extremal metrics, in particular those with "cusp" singularities at the ends. To define these metrics of Poincaré type along the lines of [3] by Auvray, who has outlined much of the recent theory of these metrics after initial investigations by Kobayashi [12] and Tian-Yau [20], one starts with a compact

Kähler manifold (M, ω_0) with an embedded compact complex hypersurface X and works locally about it:

Definition 1. *A Kähler metric ω is of Poincaré type on $M \setminus X$ if for every point $p \in X$ there exists a coordinate neighborhood $U \subset M$ about p with defining coordinate z for X (i.e. $X \cap U = \{z = 0\}$) such that on U, ω is quasi-isometric at all orders to the metric*

$$\frac{idz \wedge d\bar{z}}{|z|^2(\ln|z|^2)^2} + \omega_0|_X.$$

Furthermore, ω is said to be in the class $[\omega_0]$ if $\omega = \omega_0 + dd^c\phi$ where ϕ and all of its derivatives are bounded w.r.t ω.

In general, one would ideally be able to find criterion to determine the existence of extremal or cscK metrics, and while progress in this direction has been made for compact manifolds, such as in Chen-Cheng [6], overall the problem seems not immediately tractable. However, one can gain perspective on extremal manifolds by studying their openness and closedness properties. For instance, LeBrun-Simanca [14] showed that small deformations of an extremal Kähler class maintain the existence of extremal metrics for compact manifolds. A similar result for complete metrics in a suitably defined class are a long term goal of my research, but for the Poincaré-type metrics, Auvray has already outlined in [2] an obstruction: one can obtain an extremal (resp. cscK) Poincaré-type metric in a given class only if the divisor itself contains an extremal (resp. cscK) metric in the correct Kähler class. This implies that a broader class of metrics is required to prove perturbation results like the above, which led to our definition of a *gnarled* Poincaré-type metric, given here in the product case $\mathbb{D}^* \times X$ for simplicity:

Definition 2. *Let $\tau : \mathbb{D}^* \to \mathbb{R}$ be the map $z \mapsto \ln(-\ln|z|^2)$ so that $dd^c\tau$ is the standard cusp metric on $\mathbb{D}^*$, and take V to be a holomorphic vector field on the Kähler manifold (X, σ) with associated 1-parameter flow F_t. Then $\psi_{\tau V}$ is the gnarl associated to V where $dd^c\psi_{tV} = F_t^*\sigma - \sigma$. Then we define the gnarling of the Poincaré-type metric $\omega = dd^c\tau + \sigma$ to be the metric $\omega_V = \omega + dd^c\psi_{\tau V}$.*

Many properties of Poincaré-type metrics apply to these gnarled metrics, such as having finite volume, typically assuming V is not too large, but they are indeed not Poincaré-type metrics themselves, as the gnarling term may have exponential growth in terms of τ. With care and the introduction of a bump function in a neighborhood of X, these gnarled metrics can be defined in the $M \setminus X$ setting, named as such due to the ever more dramatic flowing experienced by a tubular neighborhood of X as you approach infinity, like the whorling gnarls seen in tree trunks.

The utility of the gnarl is highlighted when one chooses V to have a holomorphy potential, i.e. V is the gradient of some function f, as it is precisely these types of vector fields that create the aforementioned obstructions. When X is assumed to be cscK, any such f can be taken to be real-valued, and the geometry is more well-behaved. Indeed, we can then use the gnarl to prove a fundamental local version of the LeBrun-Simanca result for cscK metrics:

Theorem 1. *Let (X, σ) be cscK, then there exists a neighborhood $U \subset \mathcal{H}^{1,1}(X)$ containing 0 such that for every $\eta \in U$ there is a cscK gnarled metric ω_{V_η} on $\mathbb{D}^* \times X$ associated to a holomorphic gradient vector field V_η such that for every $z \in \mathbb{D}^*$, $[\omega_{V_\eta}]$ restricts to the class $[\sigma + \eta]$ on $\{z\} \times X$.*

This local result is the foundation for using these gnarled metrics on $M \setminus X$, but it does depend on the metric being cscK. Certainly it would be desirable to achieve a similar property in the extremal case, perhaps even starting at a gnarled metric, to get real "openness." Regardless, using the appropriate Cheng-Yau [9] weighted spaces which take into account the flowing from V, the local result has potential to be extended to the entire complete manifold $M \setminus X$, though the lack of a holomorphic tubular neighborhood about X in general presents technical complications in identifying appropriate quasicoordinate neighborhoods. However, there are still special cases of interest, such as genuine products of manifolds and holomorphic fibrations. The primary goal of my thesis is to prove the following result:

Problem 1. *Let (X, σ) be a cscK hypersurface in (M, ω_0) with Aut(M) finite and suppose X has a holomorphic tubular neighborhood. If $M \setminus X$ admits a cscK PT metric $\omega \in [\omega_0]$ and $\eta \in \mathcal{H}^{1,1}(M)$ is small enough, then there exists a cscK gnarled PT metric in $[\omega_0 + \eta]$ on $M \setminus X$.*

Further Directions

A perturbation problem directly related to my thesis work is to determine if extremality (or simply being cscK) is preserved by blowing up points or submanifolds. In the former case on compact manifolds, given the right hypotheses concerning the Hamiltonian and holomorphic vector fields, Arrezo-Pacard-Singer answered in [1] in the affirmative. In lower codimension, Seyyedali-Székelyhidi furthermore answer affirmatively in [16] given similar prerequisites, so long as the codimension of the submanifold is greater than 2. In [15], Sektnan generalizes the APS result on blowing up points away from the ends, but with a limitation coming from the extremal vector field on the divisor. Since the gnarling process was developed to handle this constraint, I am interested in using gnarled metrics to remove that limitation and attempting to show that extremality holds even when blowing up submanifolds.

Question 1. *Does the suitably behaved blowup of an extremal Poincaré-type manifold along a submanifold admit extremal gnarled metrics, and what occurs when the blowup locus intersects the defining divisor for the Poincaré-type metric?*

Notions of closedness properties of extremal metrics also remain largely uncodified, yet they may provide examples of the complete extremal metrics I have been working on. The aforementioned Calabi functional has as gradient flow since dubbed the Calabi flow given by

$$\frac{\partial}{\partial t}\omega_t = dd^c \operatorname{Scal}(\omega_t).$$

While important preliminary work has been done by Chen-He as in [8], the hope would be that carrying out the Calabi flow within an extremal class starting at a generic metric would bring about convergence to an extremal metric. However, when running the flow in a non-extremal class, one can encounter a fracturing of the starting manifold into different non-compact but complete extremal components, as observed by Székelyhidi in [18]. This leads back to queries in terms of complete extremal metrics, as it would be illuminating to know what sort of metrics actually arise from this flowing process, or in taking limits of extremal, cscK, or even Kähler-Einstein manifolds as Song has investigated recently in [17]. While my work on developing gnarled metrics has shown them to be a useful tool in handling the inhibitions of Poincaré-type metrics as previously defined, it would be remarkable to see them arise naturally in one of these other contexts.

Question 2. *What kinds of metrics/geometries arise when taking sequences of extremal (or cscK or KE) manifolds, and are Poincaré-type or gnarled Poincaré-type metrics among them?*

Moving more broadly than my thesis work, my interests can be described as developing the Calabi extremal metrics project in more expansive contexts than for which it was originally proposed. As delineated above, I would very much like to develop a coherent theory of complete extremal metrics including an appropriate conception of Kähler class. However, moving to non-compact manifolds is only one way to move outside of the initial paradigm. Relaxing the Kähler condition has led to various interesting results and opens the scope up to potentially all complex manifolds or even to almost complex manifolds and generalized complex manifolds. Among such programs are the various results regarding resolutions of Calabi Conjecture-like Monge-Ampère problems for Gauduchon metrics as explored by Székelyhidi-Tosatti-Weinkove in [19] and symplectic forms by Tosatti-Weinkove-Yau in [21]. Furthermore, questions still remain unanswered regarding the Chern-Scalar curvature for various sorts of Hermitian metrics as well as the best way to include the different cohomology theories that exist on complex and almost-complex manifolds with respect to such metric considerations.

Question 3. *What can be said of Hermitian metrics, or even almost Hermitian metrics, which satisfy a corresponding "extremal" differential equation regarding their scalar (or Chern-Scalar) curvatures having gradients whose flows preserve the almost complex structure?*

Finally, of great recent interest is the Generalized Ricci Flow ansatz introduced by Garcia-Fernandez and Streets in [10], as well as other sorts of flows proposed for Hermitian manifolds. As the gnarling construction and many of the results therein rely on flows along a vector field, such geometric flows feel familiar and inviting to me. So many of the ideas in this direction rely on Hitchin's and Gualtieri's [11] development of generalized complex geometry, thus it would be an interesting project to find methods bridging CR geometry and contact geometry along the same lines that said authors have expounded upon a spectrum of geometries between symplectic geometry and complex geometry, which could then have broad application to several branches of differential geometry.

References

[1] Arezzo C.; Pacard, F.; Singer, M. A. "Extremal metrics on blow ups." *Duke Math. J.*, (2011) 157(1):1–51.

[2] Auvray, H. "Asymptotic properties of extremal Kähler metrics of Poincaré type." 2014. arXiv:1401.0123, to appear in *Proceedings of the London Mathematical Society*

[3] Auvray, H. "The space of Poincaré type Kähler metrics on the complement of a divisor." *J. Reine Angew. Math.* (2017) 722:1–64.

[4] Burns, D.; de Bartolomeis,, P. "Stability of vector bundles and extremal metrics." *Inv. Math.* 92 (1988), 403-407.

[5] Calabi, E. "Extremal Kähler metrics." In *Seminar on Differential Geometry*, S. T. Yau, editor. Princeton, (1982).

[6] Chen, X. X.; Cheng, J. "On the constant scalar curvature Kähler metrics, apriori estimates," arXiv:1712.06697

[7] Chen, X. X.; Donaldson, S. K.; Sun, S. "Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches 2π and completion of the main proof." *J. Amer. Math. Soc.* 28 (2015), 235-278.

[8] Chen, X. X.; He, W. Y. "On the Calabi Flow." *American Journal of Mathematics* 130, no. 2 (2008): 539-570.

[9] Cheng, S. - Y.; Yau, S. - T. "Inequality between Chern numbers of singular Kähler surfaces and characterization of orbit space of discrete group of SU(2, 1)." *Contemporary Mathematics* 49 (1986), 31-43.

[10] Garcia-Fernandez, M.; Streets, J. "Generalized Ricci Flow." arXiv:2008.07004 [math.DG]

[11] Gualtieri, M. "Generalized complex geometry." *Ann. of Math.* (2) 174 (2011), no. 1, 75–123.

[12] Kobayashi, R. "Kähler-Einstein metric on an open algebraic manifold." *Osaka J. Math.* 21 (2) (1984) 399 - 418.

[13] Kodaira, K. "On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties)." *Annals of Mathematics*, Second Series, 60 (1954), 28-48.

[14] LeBrun, C.; Simanca, S. R. "On the Kähler classes of extremal metrics." In *Geometry and global analysis* (Sendai, 1993), pages 255–271. Tohoku Univ.

[15] Sektnan, L. M. "Blowing up extremal Poincaré type manifolds." arXiv:1811.12584 2018

[16] Seyyedali, R.; Székelyhidi, G. "Extremal metrics on blowups along submanifolds." *J. Differential Geom.* 114 (1) 171 - 192, January 2020.

[17] Song, J. "Degeneration of Kahler-Einstein manifolds of negative scalar curvature." arXiv:1706.01518v1

[18] Székelyhidi, G. "The Calabi functional on a ruled surface." *Annales scientifiques de l'École Normale Supérieure*, Série 4, Tome 42 (2009) no. 5, pp. 837-856.

[19] Székelyhidi, G.; Tosatti, V; Weinkove, B. "Gauduchon metrics with prescribed volume form." *Acta Math.*, 219 (2017), 181-211.

[20] Tian, G.; Yau, S. T. "Complete Kahler Manifolds with Zero Ricci Curvature." *I. Journal of the American Mathematical Society,* 3(3), (1990), 579–609.

[21] Tosatti, V; Weinkove, B.; Yau, ST. "Taming symplectic forms and the Calabi–Yau equation." *Proc. London Math. Soc.* 97 (2008), no.2, 401-424.