1.3 Algorithms and Convergence

Algorithm & Pseudocode

- An **algorithm** is an ordered sequence of unambiguous and well-defined instructions that performs some tasks.
- **Pseudocode** is an artificial and informal high-level language that describes the operating principle of a computer program or algorithm.
  - Pseudocode allows ones to focus on the logic of the algorithm without being distracted by details of language syntax.
  - The pseudo-code is a "text-based" detail (algorithmic) design tool and is complete. It describes the entire logic of the algorithm so that implementation is a task of translating line by line into source code.
  - See one page summary of pseudocode in the course website

Pseudocode

Example. Compute $\sum_{i=1}^{N} x_i$

INPUT $N, x_1, x_2, \ldots, x_N$.

OUTPUT $SUM = \sum_{i=1}^{N} x_i$

Step 1 Set $SUM = 0$. // Initialize accumulator

Step 2 For $i = 1, 2, \ldots N$ do

set $SUM = SUM + x_i$. // add next term

Step 3 OUTPUT(SUM);

STOP.

Pseudocode→ Matlab code

Example. Compute $\sum_{i=1}^{N} x_i$

```
-------------------- mySum.m -------------------------------
function [SUM] = mySum(N, X)
% INPUT:    N, X (vector of length N)
% OUTPUT:   SUM

SUM=0;      % STEP 1
for i=1:N   % STEP 2
    SUM = SUM + X(i);
end
return
------------------------------------------------------------------
```

%% compute the sum $\sum_{i=1}^{10} i$ (=55)

```
>> N  = 10; X = [1:N]
>> SUM = mySum(N,X)
```

Characterizing Algorithms

Error Growth

Suppose $E_0 > 0$ denotes an initial error, and E_n is the error after n subsequent operations.

1. If $E_n \approx CnE_0$, where C is a const. independent of n: the growth of error is **linear**. (stable)
2. If $E_n \approx C^n E_0$, where $C > 1$: the growth of error is **exponential.** (unstable)

Remark: linear growth is unavoidable; exponential growth must be avoided.

Stability

- Stable algorithm: small changes in the initial data produce small changes in the final result
- Unstable or conditionally stable algorithm: small changes in all or some initial data produce large errors

Definition 1.18 Rate of convergence for sequences

Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence converging to 0, and $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number α. If a positive constant K exists with

$$|\alpha_n - \alpha| \leq K|\beta_n|, \qquad \text{for large } n,$$

then $\{\alpha_n\}_{n=1}^{\infty}$ is said to converges to α with rate of convergence $O(\beta_n)$, indicated by $\alpha_n = \alpha + O(\beta_n)$.

Typical $\{\beta_n\}_{n=1}^{\infty}$:

$$\beta_n = \frac{1}{n^p} \quad \text{for some } p > 0$$

Example 1. Suppose that, for $n \geq 1$, $\alpha_n = \frac{n+3}{n^3}$.

The sequence $\{\alpha_n\}_{n=1}^{\infty}$ converges to 0. Find the rate of convergence for this sequence.

Example 2. Find the rate of convergence of

$$\lim_{n\to\infty} \sin\left(\frac{1}{n}\right) = 0$$

Definition 1.19 Rate of convergence for functions

Suppose that $\lim_{h\to 0} G(h) = 0$ and $\lim_{h\to 0} F(h) = L$.
If a positive constant K exists with

$$|F(h) - L| \leq K|G(h)|, \qquad \text{for sufficiently small } h,$$

then $F(h) = L + O\big(G(h)\big)$.

Typical $G(h)$:

$$G(h) = h^P \ \text{ for some } p > 0$$

Example 3. Use the third Taylor polynomial about $h = 0$ to show that $cos h + \frac{1}{2}h^2 = 1 + \mathcal{O}(h^4)$.

Example 4. Find the rate of convergence of

$$\lim_{h\to 0} \frac{\sin(h)}{h} = 1$$