1.3 Algorithms and Convergence
Algorithm & Pseudocode

• **Algorithm** is an ordered sequence of unambiguous and well-defined instructions that performs some tasks.

• **Pseudocode** is an artificial and informal high-level language that describes the operating principle of a computer program or algorithm.
  
  – Pseudocode allows ones to focus on the logic of the algorithm without being distracted by details of language syntax.
  
  – The pseudo-code is a "text-based" detail (algorithmic) design tool and is complete. It describes the entire logic of the algorithm so that implementation is a task of translating line by line into source code.

  – See one page summary of pseudocode in the course website
Pseudocode

Example. Compute $\sum_{i=1}^{N} x_i$

INPUT $N, x_1, x_2, ..., x_N$.

OUTPUT $SUM = \sum_{i=1}^{N} x_i$

Step 1 Set $SUM = 0$. // Initialize accumulator

Step 2 For $i = 1, 2, ..., N$ do

set $SUM = SUM + x_i$. // add next term

Step 3 OUTPUT(SUM);

STOP.
Example. Compute $\sum_{i=1}^{N} x_i$

------------------------ mySum.m ------------------------

function [SUM] = mySum(N, X)

% INPUT:   N, X (vector of length N)
% OUTPUT:  SUM

SUM=0;     % STEP 1
for i=1:N  % STEP 2
    SUM = SUM + X(i);
end
return

------------------------------------------------------------------------

%% compute the sum $\sum_{i=1}^{10} i$ (=55)
>> N  = 10; X = [1:N]
>> SUM = mySum(N,X)
Characterizing Algorithms

Error Growth

Suppose \( E_0 > 0 \) denotes an initial error, and \( E_n \) is the error after \( n \) subsequent operations.

1. If \( E_n \approx CnE_0 \), where \( C \) is a const. independent of \( n \): the growth of error is **linear. (stable)**

2. If \( E_n \approx C^n E_0 \), where \( C > 1 \): the growth of error is **exponential. (unstable)**

**Remark:** linear growth is unavoidable; exponential growth must be avoided.

Stability

- **Stable algorithm:** small changes in the initial data produce small changes in the final result
- **Unstable or conditionally stable algorithm:** small changes in all or some initial data produce large errors
Definition 1.18 Rate of convergence for sequences

Suppose $\{\beta_n\}_{n=1}^{\infty}$ is a sequence converging to 0, and $\{\alpha_n\}_{n=1}^{\infty}$ converges to a number $\alpha$. If a positive constant $K$ exists with

$$|\alpha_n - \alpha| \leq K |\beta_n|,$$

for large $n$, then $\{\alpha_n\}_{n=1}^{\infty}$ is said to converges to $\alpha$ with rate of convergence $O(\beta_n)$, indicated by $\alpha_n = \alpha + O(\beta_n)$.

Typical $\{\beta_n\}_{n=1}^{\infty}$:

$$\beta_n = \frac{1}{n^p} \quad \text{for some } p > 0$$
Example 1. Suppose that, for \( n \geq 1 \), \( \alpha_n = \frac{n+3}{n^3} \).

The sequence \( \{\alpha_n\}_{n=1}^{\infty} \) converges to 0. Find the rate of convergence for this sequence.

Example 2. Find the rate of convergence of

\[
\lim_{n \to \infty} \sin \left( \frac{1}{n} \right) = 0
\]
**Definition 1.19 Rate of convergence for functions**

Suppose that \( \lim_{h \to 0} G(h) = 0 \) and \( \lim_{h \to 0} F(h) = L \). If a positive constant \( K \) exists with
\[
|F(h) - L| \leq K|G(h)|, \quad \text{for sufficiently small } h,
\]
then \( F(h) = L + O(G(h)) \).

**Typical \( G(h) \):**
\[
G(h) = h^p \quad \text{for some } p > 0
\]
Example 3. Use the third Taylor polynomial about $h = 0$ to show that $\cosh + \frac{1}{2} h^2 = 1 + \mathcal{O}(h^4)$.

Example 4. Find the rate of convergence of

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1$$